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BULLETIN
OF THE
RESEARCH COUNCIL OF ISRAEL

Bull. Res. Counc. of Israel

IN HONOUR
OF
PROF. ALBERT EINSTEIN
ON HIS 75TH BIRTHDAY

VOLUME III MARCH 1954 NUMBER 4
Issued June 1954

JERUSALEM

BULLETIN OF THE RESEARCH COUNCIL OF ISRAEL

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(cont'd on inside back cover)

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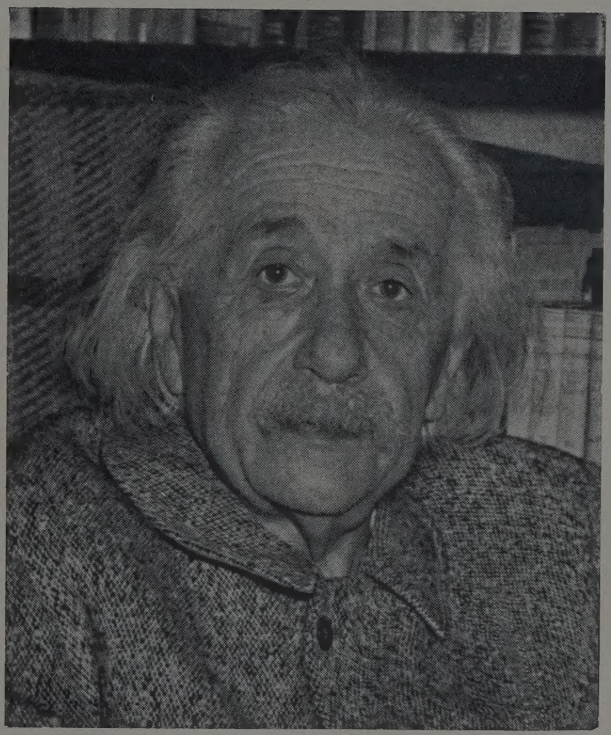
on his 75th birthday,
this issue is dedicated with
the highest esteem and affection
of the scientists of Israel



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THE INTUITIONISTIC REVOLUTION IN MATHEMATICS AND LOGIC

A. A. Fraenkel
Department of Mathematics, The Hebrew University of Jerusalem

In December 1951 I had the privilege of talking to Professor Einstein and describing the recent controversies between (neo-)intuitionists and their "formalistic" and "logicistic" antagonists; I pointed out that the first attitude would mean a kind of atomistic theory of functions, comparable to the atomistic structure of matter and energy. Einstein showed a lively interest in the subject and pointed out that to the physicist such a theory would seem by far preferable to the classical theory of continuity. I objected by stressing the main difficulty, namely, the fact that the procedures of mathematical analysis, e.g. of differential equations, are based on the assumption of mathematical continuity, while a modification sufficient to cover an intuitionistic-discrete medium cannot easily be imagined. Einstein did not share this pessimism and urged mathematicians to try to develop suitable new methods not based on continuity.

This discussion has induced me to present here a short description of the nucleus of intuitionism, particularly in its far-reaching form proposed by Brouwer and adopted by Weyl in the 1920's. This, however, does not mean taking an attitude in favour of intuitionism.

The crisis in Greek mathematics which developed in the Pythagorean school in the sixth century B. C., as well as the crisis of the foundation of calculus in the eighteenth century, were apparently solved by satisfactory methods: the former by the theory of proportions as presented in Euclid's book Five, the latter by Cauchy's doctrine of limits and by the arithmetical theories of irrational number as given by Dedekind, Weierstrass and others. However, during the last fifty years it has turned out that these methods had not succeeded in actually solving the problems involved. This disappointment became manifest in the foundations of mathematics in general and in the theory of continuum in particular.

As to the *continuum*, i. e. the set of points on a line (or on a segment) or the aggregate of real numbers, Zenon and his contemporaries of the Eleatic school, no less than medieval scholastics (whose logical achievements have only recently been valued appropriately), tried in vain to reveal the "mystery of continuum", and from Thomas Aquinas to our times we observe a reiterated readiness, born of despair, to accept continuum as a dogma or as an "original intuition". The would-be complete constructions or descriptions of the continuum achieved during the last third of the nineteenth century in real analysis on the one side and in set theory (Cantor) on the other, were attacked by H. Poincaré, B. Russell, H. Weyl and others as insufficient or meaningless because of their using vicious circles — in particular, of their defining particular objects with a certain property by the totality of those objects. At any rate, such methods are not fit

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to *construct* the continuum and to bridge the abyss between the discrete and individual realm of integers and the continuous and homogeneous medium of points in space.

At the same time, in the first decade of the twentieth century, the discussion was enlarged and transferred to the *foundations of mathematics* in general by the logical antinomies that were pointed out between 1896 and 1905, mostly connected with the concept of *set* (aggregate) which had just opened new horizons in mathematics: not only as a subtle and powerful instrument in analysis and geometry but also as an access to deeper foundations of mathematics as a whole. The first effect of the antinomies was catastrophical, so much so that a few outstanding scholars explicitly abandoned some of their results. Even after Russell had shown that the antinomies had no properly mathematical essence but were closely connected with certain logical difficulties discovered by Greek philosophers, as the Liar (Epimenides), the shock turned out to have been profound enough as to cause scepticism regarding mathematical analysis in the "classical" shape, which had been thought in the 1870's and 1880's to be satisfactory and final.

I cannot describe here the various elaborate systems which emerged from the objections specified. They vary from methods faithfully keeping to the tradition, to a revolutionary attitude which abandons the bulk of classical mathematics. Most surprising results have been obtained, in particular Gödel's incompleteness theorem of 1930 which shows that the power of deductive inference is, by its very nature, much more limited than had been assumed still in the first quarter of the present century. I shall therefore restrict myself to analysing a few characteristic lines, both mathematical and logical, of the intuitionistic trend which on the whole is the most revolutionary one: the Dutch school, founded in 1907 by L. E. J. Brouwer, who has named it neo-intuitionism. This trend should particularly be distinguished from the French school of intuitionists (Borel, Lebesgue, Lusin etc.) who do not go as far as the Dutch.

The main thesis of neo-intuitionists is that *mathematical existence coincides with construction*. According to them, neglecting this principle and identifying existence with non-contradiction has brought about not only the antinomies but classical analysis of the nineteenth century. Hence, the majority of its concepts and theorems ought to be abandoned, and the remains should be based on new foundations. Also revolutionary opinions about the relation between mathematics and language evolve from the main thesis. A second thesis, especially peculiar to Brouwer's theory of continuum, concerns the "*sequences of choice*"; in addition to sequences of rationals which are determined by a given law, sequences are also admitted here whose members are successively chosen in an arbitrary way, which implies that the sequence (representing a real number) can never be considered a finished whole. It is obvious that the intuitionistic theory of continuum essentially bears on the applicability of mathematics to physics.

The main thesis would imply the necessity of an exact definition, or at least description, of *constructibility*. It is just such a description which Brouwer has refused to give, while the definitions of others, e.g. Menger, were emphatically rejected by him. It is true that he agrees to Kronecker's and Poincaré's attitude that mathematical induction, i. e. the general concept of integer, is the main constructive source of mathematics. Beyond this, however, he is satisfied with giving a few simple instances of constructive steps, maintaining that the very dynamic character of mathematics prevents us from foreseeing all "constructive" devices possibly needed for a certain purpose —

in analogy to a mountaineer who does not in advance know all possible obstacles he has to overcome before reaching the summit.

Naturally such vagueness must not prevail in mathematical *exposition* where definitions and procedures of proof should be well determined. However, says Brouwer, mathematical exposition, static in its essence, uses the *language* to describe mathematical constructions, and language is an accessory means, serving "only" to transfer the expression of construction from one person to another, while construction in itself is an original and primitive act which cannot adequately be described by a linguistic process. For the systematization of mathematics by language, non-contradiction may be an essential factor; but by no means can it coincide with the construction or be the arbiter of legitimacy of construction — no more than the impossibility of proving a defendant's guilt signifies that he is innocent. On the other hand, mental constructions, as well as nature's processes, do not need a proof of logical compatibility.

The neo-intuitionistic doctrine of the relation between *mathematics* and *logic* then follows immediately. G. Frege, B. Russell and the logicians influenced by them maintain that mathematics is based on logic and grows from it "as boys grow to become men". On the other hand, the "formalistic" and axiomatic attitudes originating from Hilbert, while not conceiving logic as the very source of mathematics, regard it as an instrument without which mathematics can certainly not do, and at the same time as the source of "metamathematics" whose (to be sure, presumably unattainable) aim is to prove the compatibility of the formal axiomatic system that means mathematics.

In contrast with these attitudes, the Dutch school contends that logic, far from being at the bottom of mathematics or independent of it, is just derived from mathematics and has no autonomous existence. According to this view, the laws of logic are mere abstractions from mathematical constructions, and these are their only justification. As to Aristotle's logical principles, the law of identity and the law of contradiction are retained, yet the principle of the excluded middle (*tertium non datur*) is denied any autonomous validity. It may be affirmed only as an abstraction from the constructions admissible in the domain of *finite sets*, and it is senseless and unjustifiable to apply it automatically within the infinite sets prevailing all over mathematics.

Aristotle formulated the principle of the excluded middle as follows: if S denotes a given set of objects (elements), and p a property which is meaningful (not necessarily true!) for the elements of S, then *either* all elements of S have the property p, *or* one element at least of S has the contradictory property non-p. Propositions of the first kind are called general propositions, those of the second kind existential propositions. Our principle asserts that the negation of a general proposition is an existential one, and that no third possibility is left. Let us scrutinize this state of affairs in the light of four simple instances.

(a) Take the general proposition: all citizens of Israel (at a certain time) were born between the northern and the southern polar circles. Its negation therefore maintains that at least one citizen was born northward of the first or southward of the second circle. Do we really need Aristotle's principle to assert that either the first or the second proposition holds true?

Of course not. Starting from the assumption that we can ascertain the origin of each single citizen, and perusing all citizens, we either find that each of them was born between the circles, thus obtaining the general proposition, or we find an exception, which

justifies the existential proposition. In other words, our alternative results from the *conjunction* (for the general proposition) or the *disjunction* (for the existential one) of a finite number of particular propositions. And this, says the intuitionist, is the whole contents of the so-called "principle" *tertium non datur*; it is an abstract drawn from perusing particular propositions, and the so-called general proposition is just a short formulation of the conjunction: *A* has the property *p*, and *B* has it, etc. No wonder that in the second case we have a quasi-existential proposition which is also accepted by intuitionism; in fact the second result is not existential but merely the disjunction (obtained by constructions): *A* has the property non-*p*, or *B* has it, etc.

(b) Consider now the proposition: either *all* integers of the form $2^n + 1$, where n goes over all powers 2^q (q any integer surpassing 4*), are composite numbers (general proposition), or *at least one* prime number is among them (existential). Up to now no prime has been discovered among these "Gauss numbers". (Fermat had guessed that *all* numbers $2^n + 1$, where n is a power of 2, were primes; but Euler showed that, for $n=2^5$, $2^n + 1$ is composite, and the same turned out later for all those n which one has succeeded in investigating.)

Given a certain o, in principle, one can decide whether $2^n + 1$ with $n = 2^q$ is prime or not, since a finite number of steps is sufficient for the decision. However, in order to assert "*no* $q > 4$ gives a prime $2^n + 1$", we ought to peruse all the infinitely many natural numbers q that surpass 4 — as long as we have not found a general proof showing that $2^n + 1$, for such n, cannot be prime. Thus, in contrast with instance (a) where a finite aggregate only had to be examined, we cannot accomplish the infinite conjunction, hence a genuine general proposition is involved.

Negating this general proposition does not furnish us, according to intuitionism, with any kind of legitimate proposition; the sentence "there is a prime number of the form in question" is purely existential and void of meaning, so far as such a prime has not been constructed.

(c) The expansion of π into a decimal fraction is known up to thousands of digits. Heretofore, however, certain general questions about the sequence of digits as a whole have not been solved; for instance, the question of whether the digit 7 ever occurs seven successive times. According to the principle of the excluded middle, this would either happen, and then once for the first time in the expansion, defining the nth place after the decimal point as the beginning of the first sequence 7777777 — or else the expansion contains no seven successive 7's. But as long as a general proof of this impossibility has not been reached, the intuitionist would not acknowledge the disjunctive proposition just expressed; a mathematical theorem, proved both on the assumption that such a sequence occurs and on the assumption that it does not, would not be considered to be proven.

Analogically as in example (b), for each single value of $n = 1, 2, \ldots$ it can be decided by a finite procedure whether at the nth place a sequence of seven 7's begins or not. The quantification applying to all values of n, however, transforms the problem into an infinite one.

(d) Fermat's Last Theorem maintains that for no positive integer n can the equation $x^{n+2} + y^{n+2} = z^{n+2}$ be solved by natural numbers x, y, z. While for infinitely many

* For $q = 0, 1, 2, 3, 4$, one obtains prime numbers.

values of n the theorem has been proved, there remain also infinitely many cases unsettled (and no counterinstance has been discovered). In this case, the intuitionist not only denies the possibility of asserting that the theorem is either true or false, but even for a given n — except those n for which a proof has been accomplished — he would not admit the equation to be either solvable or not, because there exists no general finite method to decide between both cases.

Examples (a)—(d) show, firstly in what sense one cannot, with regard to infinite aggregates, justify the use of *tertium non datur* by the method used in (a), and secondly that the "third" case is not coordinate with the first two (general proof, construction). To be sure, for more than one generation, traditional two-valued logic (with the values true and false) has been generalized to a logic of n truth-values, n assuming integral values surpassing 1, or even infinite values (for instance, in the theory of probability). In the present case, however, as should be stressed against certain misconceptions, we are not dealing with a logic with three truth-values: true, false, third case — a logic which accordingly would require a principle of *quartum non datur;* for the third case is qualitatively distinct from the other two, being not objective and absolute but dependent on the present stage of scientific research. In each of the cases (b) — (d) the trichotomy may disappear to-morrow, owing to the achievement of a general proof or to the discovery of a counter-example. Yet this possibility does not impress the intuitionists who point out that, instead of any problem solved in this sense, a new unsolved problem can be indicated.

Now this state of affairs expresses itself not only on the borderline of mathematical research but it applies to elementary theorems of analysis, such as the proposition that two well-defined real numbers are either equal or different. Let us define, e. g., numbers a and b as follows. $a=1/3$; b is a decimal beginning also with 0.333 . . . . and continuing with the digit 3, except for the following case: if at the mth place in the decimal expansion of π for the first time a sequence of seven 7's is starting, then the mth digit of b shall be 2 if m is odd, 4 if m is even. We have to-day no means of deciding whether b is equal, greater or smaller than a; therefore Brouwer considers a and b to be incomparable — which has actual implications. For instance, the proof of the so-called fundamental theorem of algebra considers the function of the coefficients of a given equation, called: discriminant, and proceeds in different ways according as this function equals 0 or not. But a constructive decision between these two is impossible when the discriminant is divisible by $a-b$. In view of this, in 1924, three leading intuitionists (Brouwer, Skolem, Weyl) independently elaborated new proofs of the fundamental theorem, regarding the dozens of proofs given since 1799 as inadequate.

To be sure, the "classical" mathematician or logician will answer: intrinsically and objectively, only one of two possibilities can obtain! For instance, in example (b): either $2^{16}+1$ is the last prime number of the form 2^n+1, or there is one greater than $2^{16}+1$. In the latter case we are even sure in advance that a constructive finite procedure (though its length is not bounded *a priori*) will verify its truth: accordingly, the objective lack of such procedure means that the first case is true. Where is, then, room for a "third" case? The Dutch school regards this argument as void, since it relies on the assumption of absolute truth in a world of Platonic ideas, which has nothing to do with scientific truth: mathematical concepts and propositions exist only as far as they are constructed and proven by the creative activity of human mind, and this acti-

vity is their only *raison d'être*. Relying on non-contradiction or on projections upon a supposed world of objective entities, is not science but metaphysical speculation — and this, alas, is what mathematicians have been persisting in doing from the seventeenth century up to our days, with a few exceptions.

Though the number of conclusive intuitionists has remained small up to now, and though many of them (in particular those of the Paris school: Borel, Lebesgue, etc.) did not always, in their mathematical research work, refrain from using procedures which they declined in principle, the influence of those sceptical attitudes is felt more and more in wide mathematical circles. If this is not in agreeing with the decisive restriction of the mathematical domain as implied by the principles of neo-intuitionism, so at least in demanding safer and less "absolutist" foundations for classical mathematics than those recognized through the nineteenth century — and no such foundations have turned up as yet. This means an unexpected and profound revolution within two or three decades from the International Congress of Mathematicians in Paris in 1900, when Hilbert in his historical address on Mathematical Problems, under general consensus, declared: for the mathematician there is no *ignorabimus*; all of you share the conviction that each definite mathematical problem necessarily admits a strict settlement, and you are aware of the continual call addressing you "behold the problem, seek its solution"; you can find it by pure thought (using, of course, the principle of the excluded middle). What almost tragic turn from such conviction to one of the profoundest achievement of the next generation, to the proof — by as absolutist a thinker as Gödel — in 1940 that his solution of the first of Hilbert's problems, the continuum problem, is consistent, i. e. non-contradictory; or to the justification of classical arithmetic by mapping it on a subclass of intuitionistic arithmetic (cf. below).

One may say that the most interesting results in the field of intuitionism have been reached by handling intuitionistic propositions with classic methods; in particular, with the means of symbolic logic. In the 20's and the early 30's European scholars, as Heyting, Gödel, Glivenko, Kolmogoroff, Gentzen, Jaśkowski started this procedure, which was afterwards developed by Americans, e. g. Stone, Birkhoff, Kleene, etc. To illustrate the situation consider the following example which only needs one pair of symbols, that of classical negation $\sim$ and that of neo-intuitionistic negation $\neg$ (absurdity) $\neg$. The *tertium non datur* then appears under either of the forms "p or $\sim p$", "$\sim\sim p$ is equivalent p", p denoting any proposition. In the logic of neo-intuitionism this theorem is no more valid, and we have to distinguish between three cases: p, $\neg p$, $\neg\neg p$. However, if q is a negative proposition $\neg p$, asserting the absurdity of p, then the *tertium non datur* obtains for q; in other words, the equivalence between $\neg\neg\neg p$ and $\neg p$ holds generally. One can also show that if p can be proved in classical logic, $\neg\neg p$ can be proved in intuitionistic logic, and conversely; and if $\neg p$ is provable classically, the same holds intuitionistically. The system of axioms for the classical calculus of propositions is transformed to Heyting's intuitionistic system by replacing the *tertium non datur* "p or non-p" with the weaker principle: a proposition which implies p as well as $\neg p$, is absurd.

Any formula that is classically provable, is intuitionistically at least compatible (non-contradictory). Hence the intuitionistic calculus of propositions constitutes *part* of the classical calculus, and it cannot be represented by a logical system with a finite number of truth-values. In fact, an infinite sequence of "decreasing" logical systems is interspersed between the classical and the intuitionistic systems.

Perhaps the most surprising result is the following. Though it was just said that Heyting's calculus is part of the classical one, this statement needs a modification: while it is true in view of the ordinary correspondence between classical and intuitionistic concepts (e. g., between $\sim$ and $\neg$), there are other ways of correspondence between the primitive concepts of both calculi — and among them such as map the classical calculus on part of the intuitionistic one (Gödel).

To be sure, this mapping is built on the excluded middle; for this reason, as well as for the very method of mapping, it is not acceptable to intuitionists. Nevertheless it reveals a certain relativity with respect to the extension of both calculi. This relativity can even be transferred from the calculus of propositions to classical and intuitionistic arithmetics, and would then, for classical eyes, yield a proof of non-contradiction of classical arithmetics by means of intuitionistic constructions.

On the whole one has to admit that the restrictions of intuitionism imply a serious amputation on the domain and the methods of mathematics, and this is the main reason why the vast majority of mathematicians have refused to accept them. While these restrictions are necessarily still more felt by the physicist who continually applies calculus and classical analysis in general to his problems, he will consider the intuitionistic trends to contain a profound analogy to certain ideas of modern physics. In particular the intuitionistic theories of continuum, which renounce genuine mathematical continuity in favour of either "atomistic" attitudes or of the conception of continuum as a medium of "freely developing" instead of static "being", cannot fail to evoke among physicists a longing for a new mathematical analysis fit to deal with such pseudo-continua. The future will show whether this longing can be satisfied.

THE COMPETITION BETWEEN THE LOW CONFIGURATIONS IN THE SPECTRA OF THE IRON GROUP

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I. INTRODUCTION

It is stated in every standard book of spectroscopy that the 4*s*-level is lower than 3*d*, and therefore in the building up of the periodical system the 3*d*-shell is filled after the 4*s*; and the same holds for the 4*d* and the 5*s*, and for the 5*d* and the 6*s* shells. But a thorough examination of the situation shows that "there is a close competition going on between configurations of the type d^n, $d^{n-1}s$ and $d^{n-2}s^2$ for the distinction of contributing to the normal state of the atom"[1], and that $d^{n-2}s^2$ is not always the winner: on the contrary, in many elements the normal state belongs to $d^{n-1}s$, and in Pd even to d^{10}, as is shown in Table I. The situation is even more complicated in the ionized atoms, as the fundamental levels of the second spectra belong to d^n, $d^{n-1}s$, and even to $d^{n-2}s^2$ without any apparent order.

TABLE I

Number of s-electrons in the ground states of atoms and ions in the first two long periods

Element	K	Ca	Sc	Ti	V	Cr	Mn	Fe	Co	Ni	Cu	Zn
Neutral Atom	1	2	2	2	2	1	2	2	2	2	1	2
Ionized Atom	0	1	1	1	0	0	1	1	0	0	0	1

Element	Rb	Sr	Y	Zr	Nb	Mo	Tc	Ru	Rh	Pd	Ag	Cd
Neutral Atom	1	2	2	2	1	1	2	1	1	0	1	2
Ionized Atom	0	1	2	1	0	0	1	0	0	0	0	1

A first attempt at a quantitative comparison of the positions of the different low configurations in the spectra of the iron group was made by Russell[2] in his classical paper of 1927.

Since then both our experimental knowledge and our theoretical understanding of these spectra have enormously increased, and a new and very successful comparison has now been made by Catalan, Rohrlich and Shenstone (CRS)[3].

These authors break with the old-established tradition of comparing sequences of isoelectronic spectra with different degrees of ionization; instead, they consider the relative positions of the lowest terms of the configurations d^n, $d^{n-1}s$ and $d^{n-2}s^2$ for a fixed ionization degree. Their results are represented as functions of n by very broken lines (Figures 1 and 2 of CRS), as could be expected from the apparent disorder of Table I; but these broken lines show very striking regularities, which are repeated almost exactly for the different degrees of ionization and for the different long periods.

The form of these diagrams has already been discussed and compared with theory by CRS; but it is perhaps worthwhile to discuss them from a more general point of view,

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by substituting their Tables, III and IV, of theoretical results with closed formulas, which allow a better understanding of the physical meaning of the regularities and irregularities of the diagrams. Furthermore, by comparing baricenters of groups of terms instead of individual terms, we shall obtain simpler interpolation formulas which may be useful for the prediction of still unknown terms, and we shall see how the method may be extended in a very natural way to more complex configurations.

II. THEORY

The electrostatic interaction between the electrons of a given configuration is usually expressed in terms of the Slater-Condon parameters F_k. In a previous investigation[4] it was empirically found that the formulas for the configurations d^n are simplified by substituting the F_k with some linear combinations of them; but only later[5] a rational method was found for the choice of the most convenient linear combinations. These new combinations are especially convenient for such general considerations as those contained in the present paper; since CRS used the old combinations A, B and C, we shall give the new combinations both in terms of the F^k and of A, B and C:

$$\begin{aligned} E^0 &= F^0-(F^2+F^4)/14 = A-7B/2 \\ E^1 &= 5(F^2+F^4)/98 = C+5B/2 \\ E^2 &= F^2/98-5F^4/882 = B/2. \end{aligned} \tag{1}$$

In terms of these new parameters the interaction energies of the configuration d^2 are:

$$\begin{aligned} {}^1S &= E^0+7E^1 \\ {}^3P &= E^0 \quad +21E^2 \\ {}^1D &= E^0+2E^1-9E^2 \\ {}^3F &= E^0 \quad -9E^2 \\ {}^1G &= E^0+2E^1+5E^2. \end{aligned} \tag{2}$$

The choice of E^0, E^1 and E^2 was based on group-theoretical considerations; but also a more physical definition of them may be given, which is important for the purposes of the present paper. E^0 is the baricenter of the triplets and is therefore the mean value of the interaction of two d-electrons with parallel spins; $2E^1$ is the distance between the baricenter of ${}^1D$ and ${}^1G$ and the baricenter of the triplets, and $7E^1$ is the distance of ${}^1S$ from the baricenter of the triplets; the distances of the individual terms from the baricenter of their group are expressed by means of E^2.

The terms of d^n with parallel spins may be calculated directly, without the use of Slater's diagonal-sum method, by simple vectorial considerations[6]. If the energy of the triplets of d^2 is written in the form

$$W(d^2) = E^0-9E^2-6(l_1\cdot l_2)E^2 \qquad (S=1) \tag{3}$$

we see immediately that for configurations d^n the energy is

$$\begin{aligned} W(d^n) &= {}^1/_2 n(n-1)(E^0-9E^2)-6E^2 \sum_{i<k}(l_i\cdot l_k) = \\ &= {}^1/_2 n(n-1)(E^0-9E^2)-3[L(L+1)-nl(l+1)]E^2 = \\ &= {}^1/_2 n(n-1)E^0+3[{}^3/_2 n(5-n)-L(L+1)]E^2 \qquad (S=n/2). \end{aligned} \tag{4}$$

If we consider the values of L allowed by the Pauli exclusion principle for the terms with highest multiplicity, we see that the coefficient of E^2 vanishes for n=0, 1, 4, 5, i.e. for the configurations with one term of highest multiplicity, S or D, and is different from zero for n=2, 3, i.e. for the configurations where the terms with highest multiplicity are F and P. We may therefore write instead of (4)

$$W(d^n) = {}^1/_2 n(n-1)E^0 + x(L)E^2 \qquad (S=n/2) \tag{5}$$

with

$$x(S) = x(D) = 0, \qquad x(F) = -9, \qquad x(P) = 21. \tag{6}$$

We may also remark that although $x(L)$ does not vanish in the configurations with two terms of highest multiplicity, also in these configurations

$$\Sigma_L(2L+1)\, x(L) = 0, \tag{7}$$

and therefore the baricenter of the terms with parallel spins is always given by ${}^1/_2 n(n-1)E^0$. And this is not a surprise, considering the definition of E^0 and E^2.

Equation (5) has a meaning only for $n \leqslant 5$, as for $n > 5$ there are no terms with $S = n/2$; but for $n > 5$ the electrostatic interaction may be calculated according to Chap. XIII of TAS. In particular equation $1^{13}2$ shows that $W(d^{10-\varepsilon})$ may be derived from $W(d^\varepsilon)$ by adding to it $(5-\varepsilon)$ times the expression $9^6 11$ with $nl-n'l'$. Since this last expression has the value $9E^0 + 7E^1$, we have

$$W(d^n) = W(d^{10-n}) + (n-5)(9E^0 + 7E^1) =$$

$$= {}^1/_2 n(n-1)E^0 + 7(n-5)E^1 + x(L)E^2 \qquad (S=(10-n)/2). \tag{8}$$

Equations (5) and (8) may be collected together by writing

$$W_h(d^n) = {}^1/_2 n(n-1)E^0 + {}^7/_2(n-2S)E^1 + x(L)E^2, \tag{9}$$

where the suffix h means "highest multiplicity".

The electrostatic interaction for the configurations $d^{n-1}\, s$ may be calculated according to Van Vleck[7] by means of Dirac's vector model; his result reads

$$W(d^{n-1}s) = W(d^{n-1}) + (n-1)F^0 - {}^1/_2\left\{n-1+2[S_1(S_1+1) - S_k(S_k+1) - {}^3/_4]\right\} G_2, \tag{10}$$

where S_1 and S_k, F^0 and G_2, are, respectively, the spins of the term and of the core, $F^0(ds)$ and $G_2(ds)$. For the terms with highest multiplicity $S_k = S_1 - {}^1/_2$, and (10) reduces to

$$W_h(d^{n-1}s) = W_h(d^{n-1}) + (n-1)F_0 - {}^1/_2(n+2S_1-2)G_2 =$$

$$= {}^1/_2(n-1)(n-2)E^0 + (n-1)F^0 - {}^1/_2(n+2S_1-2)G_2 + {}^7/_2(n-2S_1)E^1 + x(L_1)E^2. \tag{11}$$

The electrostatic interaction for the configuration s^2 is $F'^0 = F_0(ss)$, and the interaction between d^{n-2} and s^2 may be calculated by means of $9^6 11$ of TAS; the result is

$$W(d^{n-2}s^2) = W(d^{n-2}) + F'^0 + (n-2)(2F_0 - G_2) \tag{12}$$

and therefore

$$W_h(d^{n-2}\,s^2) = {}^1/_2(n-2)(n-3)E^0 + F'_0 + (n-2)(2F^0-G_2) + {}^7/_2(n-2S_2-2)E^1 + x(L_2)E^2. \quad (13)$$

Before we compare the energies of the different terms, we need still to add the energies of the electrons in the field of the nucleus and of the closed inner shells:

$$E(d^{n-k}\,s^k) = (n-k)\,A_d + kA_s + W(d^{n-k}s^k). \quad (14)$$

The relative positions of the lowest terms of the different configurations may now be obtained easily from (14), (9), (11) and (13):

$$E_h(d^{n-1}\,s) - E_h(d^n) = A_s - A_d - 7(S_1 - S)E^1 + \\ + (n-1)(F^0 - E^0 - G_2) + {}^1/_2(n-2S_1)G_2 + \\ + [x(L_1) - x(L)]E^2, \quad (15)$$

$$E_h(d^{n-2}\,s^2) - E_h(d^{n-1}\,s) = A_s - A_d + F'_0 - F^0 + G_2 - 7(S_2 - S_1 + 1)E^1 + \\ + (n-2)(F^0 - E^0) - {}^1/_2(n-2S_1)G_2 + \\ + [x(L_2) - x(L_1)]E^2. \quad (16)$$

The expressions (15) and (16) consist essentially of three parts: the first part is independent of n, but has a strong discontinuity in the middle of the group, as $S_1 - S$ and $S_2 - S_1 + 1$ vanish on the left side of the periodic table and equal one on the right side; the second part increases linearly with n, and also the coefficient of n has a discontinuity (but much smaller) in the middle of the group, as ${}^1/_2(n-2S_1)$ vanishes on the left side and equals $n-6$ on the right side; the last part is an oscillating function of n, as $x(L_1) - x(L)$ and $x(L_2) - x(L_1)$ take the values 0, 9, 0, -9, 0, on the left side, and again the same values on the right side, if we take always the L's of the lowest terms. We conclude that the behaviour of (15) and (16) as functions of n describes the form of the experimental diagrams (Figures 1 and 2 of CRS) very well qualitatively.

III. COMPARISON WITH EXPERIMENT

In order to make a quantitative comparison between theory and experiment, it is convenient to get rid of the oscillating term. We have already seen that the mean (weighted) value of $x(L)$ vanishes for the terms with highest multiplicity of every configuration. Therefore, if instead of comparing the energies of the lowest terms of every configuration we compare the baricenters B_h of the terms of highest multiplicity, we shall obtain the same expressions (15) and (16), but without the last term; and the ten points representing one of these differences as functions of n are expected to lie in two groups of five on two straight lines.

In the calculation of the experimental values of the baricenters, a minor difficulty arises from the fact that in some configuration only the fundamental F-term is known, and the P-term has not yet been identified. This difficulty may be overcome by remarking that the difference $P - F$ is always $30E^2$, and should therefore be a regular function of the atomic number and of the degree of ionization. Indeed an inspection of the known terms[8] in the configurations $d^{n-k}s^k$ of the iron group shows that 34 differences $P - F$ may be represented by the interpolation formula

$$P-F = 2679+3158q-111q^2+853n+1334k, \qquad (17)$$

where q is the degree of ionization; the five numerical coefficients were obtained by least squares from the experimental data* and the agreement is shown in Table II; the mean error of ± 340 seems to be very satisfactory.

TABLE II

The interpolation of the differences P-F *with highest multiplicity in the configurations* $d^{n-k}s^k$ *of the iron group*

Config.	*Spectrum*	*Obs.*	*Calc.*	*Diff.*
d^2	Ca I	5057	4385	+672
	Sc II	7219	7432	−213
	Ti III	10419	10257	+162
	V IV	12925	12860	+ 65
	Cr V	15206	15241	− 35
	Mn VI	17380	17400	− 20
	Fe VII	19509	19337	+172
d^3	Sc I	(2694)	5238	=
	Ti II	8367	8285	+ 82
	V III	11332	11110	+222
	Cr IV	13770	13713	+ 57
	Mn V	15968	16094	−126
	Fe VI	18075	18253	−178
	Co VII	20129	20190	− 61
	Ni VIII	21945	21905	+ 40
d^7	Mn I	=	8650	=
	Fe II	11195	11697	−502
	Co III	14561	14522	+ 39
d^8	Fe I	=	9503	=
	Co II	12649	12550	+ 99
	Ni III	15836	15375	+461
d^2s	Sc I	6359	6572	−213
	Ti II	9743	9619	+124
	V III	=	12444	=
	Cr IV	=	15047	=
	Mn V	=	17428	=
d^3s	Ti I	7334	7425	− 91
	V II	10721	10472	+249
	Cr III	13241	13297	− 56
	Mn IV	=	15900	=
	Fe V	=	18281	=
d^7s^1	Fe I	10225	10837	−612
	Co II	13818	13884	− 66
	Ni III	16494	16709	−215
d^8s	Co I	11391	11690	−299
	Ni II	14601	14737	−136
	Cu III	17050	17562	−512
d^2s^2	Ti I	8324	8759	−435
	V II	=	11806	=
d^3s^2	V I	9396	9612	−216
	Cr II	(1501)	12659	=
d^7s^2	Co I	13183	13024	+159
	Ni II	=	16071	=
d^8s^2	Ni I	14725	13877	+848
	Cu II	17473	16924	+549

Only two P–F differences did not fit (17) and have been discarded: they are the differences ^4P–^4F in the configurations d^3 of ScI and d^3s^2 of CrII: in the first case it seems very probable that $3d^3\ e^4P$ is strongly perturbed by $3d4s4d\ f^4P$; in the second case the disagreement of more than 10000 K shows that c^4P cannot belong to d^3s^2, and the independent calculation of Schweizer[10] shows that it may be assigned to d^4s.

Equation (17) was used for calculating the differences P—F in the cases where one of the terms is still unknown or has been discarded; and the results are given in the same Table II; the baricenters were then obtained by adding to the F-terms 3/10 of the calculated differences, or, in the case of MnI, by detracting from the ^4P-term 7/10 of the calculated difference.

*Since the d^8s^4 P-terms of NiII and CuIII are strongly perturbed by the ^2D-terms, and even the assignments of the levels with $J=3/2$ and $J=5/2$ are somewhat doubtful[9], the baricenter of these ^4P-terms was calculated by assigning to them the lowest level with $J=5/2$ and the highest one with $J=3/2$; this choice is justified not only by the g-values and the theoretical calculations, but also by the fact that the choice of one lower and one higher level will give a value for the calculated baricenter, which should be nearer to the unperturbed position.

The differences

$$D_1 = B_h(d^{n-1}s) - B_h(d^n)$$

and

$$D_2 = B_h(d^{n-2}s^2) - B_h(d^{n-1}s)$$

are given in Tables III and IV, and are compared with the values of linear functions of n fitted by least squares. Baricenters calculated by means of (17) are marked by a star.

TABLE III

Relative positions of the baricenters of the terms with highest multiplicity in the configurations d^{n-1} s *and* d^n *of the iron group.*

Spectrum	*n*	$B_h(d^{n-1}\,s)$	$B_h(d^n)$	D_1	a_1+b_1n	*Diff.*	b_1
K I	1	000	21535	(—21535)	—26652	(5177)	
Ca I	2	20357	45011	—24654	—24433	—221	
Sc I	3	13518	35417*	—21899	—22214	+315	2219
Ti I	4	8921	28881	—19960	—19995	+ 35	
V I	5	2297	20202	—17905	—17776	—129	
Cr I	6	000	35498	—35498	—34950	—548	
Mn I	7	17301	45635*	—28334	—29848	+1514	
Fe I	8	10527	36117*	—25590	—24746	—844	5102
Co I	9	7575	27886	—20311	—19644	—667	
Ni I	10	731	14279	—13998	—14542	+544	
Ca II	1	000	13687	—13687	—13546	—141	
Sc II	2	105	7075	— 6970	— 7062	+ 92	
Ti II	3	3148	3595	— 447	— 578	+131	6484
V II	4	6142	206	+ 5936	+ 5906	+ 30	
Cr II	5	12278	000	+12278	+12390	—112	
Mn II	6	000	14586	—14586	—14663	+ 77	
Fe II	7	416	5775	— 5359	— 5367	+ 8	
Co II	8	8299	4492	+ 3807	+ 3929	—122	9296
Ni II	9	13735	603	+13132	+13225	— 93	
Cu II	10	22649	000	+22649	+22521	+128	
Sc III	1	25537	119	25418	25493	— 75	
Ti III	2	38277	3368	34909	34872	+ 37	
V III	3	48076*	3736	44340	44251	+ 89	9379
Cr III	4	53992	350	53642	53630	+ 12	
Mn III	5	62948	000	62948	63009	— 61	
Fe III	6	30089	423	29666	29541	+125	
Co III	7	47047	5186	41861	41871	— 10	
Ni III	8	59782	5745	54037	54201	—164	12330
Cu III	9	67221	829	66392	66531	—139	
Zn III	10	79048	000	79048	78861	+187	
Ti IV	1	80379	231	80148	80144	+ 4	
V IV	2	96547	4297	92250	92280	— 30	12136
Cr IV	3	109151*	4685	104466	104416	+ 50	
Mn IV	4	117066*	537	116529	116552	— 23	
V V	1	148100	372	147728	147701	+ 27	
Cr V	2	167715	5224	162491	162563	— 72	14862
Mn V	3	183090*	5603	177487	177425	+ 62	
Fe V	4	193057*	789	192268	192287	— 19	

* *P—F* taken from Table II.

TABLE IV

Relative positions of the baricenters of the terms with highest multiplicity in the configurations $d^{n-2}s^2$ *and* $d^{n-1}s$ *of the iron group.*

Spectrum	*n*	B_h $(d^{n-2}s^2)$	$B_h(d^{n-1}s)$	D_2	a_2+b_2n	*Diff.*	b_2
Ca I	2	000	20357	—20357	—20439	+ 82	
Sc I	3	101	13518	—13417	—13324	— 93	
Ti I	4	2720	8921	— 6201	— 6209	+ 8	7115
V I	5	3138	2297	+ 841	+ 906	— 65	
Cr I	6	8090	000	+ 8090	+ 8021	+ 69	
Mn I	7	000	17301	—17301	—17399	+ 98	
Fe I	8	403	10527	—10124	—10057	— 67	
Co I	9	4748	7575	— 2827	— 2715	—112	7342
Ni I	10	5389	731	+ 4658	+ 4627	+ 31	
Cu I	11	12020	000	+12020	+11969	+ 51	
Sc II	2	11736	105	11631	11388	+243	
Ti II	3	25100	3148	21952	22447	—495	
V II	4	=	6142	=	33506	=	11059
Cr II	5	57355*	12278	45077	44565	+512	
Mn II	6	55364	000	55364	55624	—260	
Fe II	7	23318	416	22902	22843	+ 59	
Co II	8	41293	8299	32994	33088	— 94	
Ni II	9	=	13735	=	43333	=	10245
Cu II	10	76275	22649	53626	53578	+ 48	
Zn II	11	63810	000	63810	63823	— 13	

* *P—F* taken from Table II.

The differences between the two lowest configurations of each spectrum show an excellent agreement, as the mean quadratic deviation is only $87K$ and the mean error $115K$. In the spectra where the lowest terms of all three configurations are known, the differences involving the highest configuration show a less regular behaviour, which indicates that this last configuration is more perturbed; and this fact is very easy to understand.

IV. DISCUSSION OF THE RESULTS

The excellent agreement for the lowest configurations should not be overestimated and should be considered in some measure as casual, since the theory was oversimplified. In effect the linear dependence of D_1 and D_2 on n follows from the assumption that the parameters appearing in the first two rows of Eqs. (15) and (16) are independent of n, and this is surely not the case. The empirical Eq. (17) shows that E^2 varies linearly with n, and it is reasonable to assume that the same holds for E^1, G_2 and F^0-E^0. Then in order to explain the linearity of the D's shown in Tables III and IV, we need to assume that A_s-A_d is to a good approximation a quadratic function of n, and that its quadratic term compensates by chance the other quadratic term which arises from the linear variation of F^0-E^0. According to this interpretation the values of b given in Tables III and IV are the constant part of F^0-E^0 or of F^0-E^0 G_2 plus the coefficient of the linear variation of the first rows of (15) and (16).

However the variations of the parameters b when passing from the first half to the second half of the group are difficult to understand. According to (15) and (16), b_1 should increase by G_2, and b_2 should decrease by the same amount; instead of that, b_1 increases by about $2900K$, and b_2 remains almost unchanged. We may therefore say that in both cases b increases by about $1500K$ more than expected; and this difference

is even bigger if we consider that the linear dependence of E^1 on n should produce a decrease of b in the second half of the group.

A confirmation of our cautious interpretation of the linear dependence of the D's on the number of electrons is given by the fact that in the second long period, the D's are represented much better by quadratic functions of n than by linear functions. The significance of a least-squares interpolation of the experimental values by quadratic functions is perhaps questionable, as we have three free parameters for every group of at most five points; but an evaluation of the significance and of the importance of the quadratic terms may be obtained by considering the second differences of the experimental values, i.e. the quantities $D(n)-2D(n-1)+D(n-2)$. The mean value of these second differences* is 30 ± 36 in the iron group and 723 ± 100 in the Pd-group, thus confirming that the compensation of the quadratic terms takes place in the first long period, but not in the second.

In the third long period very few data are available, and also the deviations from LS-coupling are very strong; it is however worthwhile to remark that the four values of D_1 for the second spectra of Lu, Hf, Ta and W give two second differences, 581 and 563, which are almost equal; if we assume that the same second difference should be obtained from Yb, Lu and Hf, we may predict that the term $4f^{14}5d\ ^2D$ of YbII, which is still unknown, should lie about $24500K$ above the ground state $4f^{14}5s\ ^2S$.

It is also interesting to consider the dependence of the parameters on the degree of ionization. The dependence of E^2 is given by (17): we may say that it is a linear dependence with a quadratic correction, which is small indeed, but surely outside the statistical error (without this correction, the mean error would increase from 340 to 540).

The values of b_1 obtained in Table III show for the spark spectra an almost linear dependence on the degree of ionization; but the values for the arc spectra are about $1200K$ lower than what could be expected from an extrapolation from the spark spectra. This irregularity would be very surprising if b_1 were interpreted as the coefficient of the second row of (15), but we have already seen that b_1 contains also the linear dependence of A_s-A_d on n, and we have no reason for assuming that this difference, or even its derivative with respect to n, is a simple function of the degree of ionization.

The dependence of a_1 on the degree of ionization is still more complicated, and the data contained in Table III are not even sufficient for giving a clear idea of its behaviour. In order to have a more complete picture of the situation it is convenient to consider instead of a_1 the experimental values of D_1 for the KI isoelectronic sequence; these values are equivalent to a_1+b_1 and are known for two more spectra than reported in Table III. All the known values of this sequence are given in Table V together with their successive differences; we see from this table that the second differences are still very big, but the third differences become very small for increasing ionization.

TABLE V

The difference $4s\ ^2S-3d\ ^2D$ *in the K I isoelectronic sequence*

Spectrum	K I	Ca II	Sc III	Ti IV	V V	Cr VI	Mn VII
$^2S-{}^2D$	—21535	—13687	25418	80148	147728	227201	317921
First differences	7848	39105	54730	67580	79473	90720	
Second differences	31257	15625	12850	11893	11247		
Third differences	—15632	—2775	—957	—646			

* We considered only the second differences of D_2 for the arc spectra and only those of D_1 for the spark spectra.

This fact explains the agreements and disagreements obtained by Trees[11], who calculated the values of $W(d^4\,{}^5D)-W(d^3s\,{}^5F)$ by the "second approximation" formula of Bacher and Goudsmit[12]. Since this formula is more or less equivalent to a linear extrapolation of the interaction parameters and a quadratic extrapolation of the one-electron parameters, the expected agreement is of the order of the third differences of Table V. But if instead of extrapolating from the data of the spectra of the same atom in higher degrees of ionization, we use the same formulas of Bacher and Goudsmit for extrapolating from the data of the spectra of the preceding atoms with the same degree of ionization, the expected agreement will be of the order of the agreements of Table III. Both expectations are confirmed by the results given by Trees in his Tables I and II.

V. APPLICATIONS TO OTHER CONFIGURATIONS

The methods developed in this paper may be used almost without modification for comparing the relative positions of terms and baricenters of groups of terms in other configurations and in other sequences of the periodic table.

For example the distances between the baricenters of the terms with highest multiplicity in the configurations $3d^{n-1}4s4p$ and $3d^n\,4p$ of the first spectra of the iron group show an almost linear dependence on n with a small quadratic correction.

But the field where these methods will have the most important application is the analysis of the spectra of the rare earths. The data available in this field are in general too few for doing systematic comparisons; but the data on the second spectra of the rare earths are already sufficient for allowing reliable predictions on the relative positions of the different configurations. The linear dependence on n of the difference $f^n-f^{n-1}s$ was already pointed out by Albertson, Harrison and McNally[13]; and "a premature addition of f type electrons to fill or half-fill the $4f$-shell" was pointed out by Meggers[14].

Also apparent irregularities in the filling up of the nuclear subshells in heavy nuclei may perhaps be explained by considerations similar to those contained in the present paper.

This paper originates from discussions with Professors A. G. Shenstone and M. Catalan during the spring of 1951, when the author was a member of the Institute for Advanced Study; he takes this opportunity for thanking again Professor J. R. Oppenheimer for the generous hospitality in the stimulatiug atmosphere of Princeton.

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MAGNETIC MOMENTS OF ODD NUCLEI IN *jj* COUPLING

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INTRODUCTION

In a paper on the *jj* coupling shell model, Umesawa[1] calculated magnetic moments of nuclei up to Mn^{55}, for every nucleus separately, on the assumption that each particle outside the core possesses a magnetic moment equal to the one particle value as used in the Schmidt diagram, but combines all such particles, both protons and neutrons, into states classified by the isotopic spin T, resultant angular momentum J, reduced seniority s and reduced isotopic spin[2] t. Flowers[3], on the other hand, according to the same assumption, has also obtained a formula of magnetic moments for three particles outside the core, in the configuration j^3, in the ground state, with isotopic spin $T=1/2$ and seniority $s=1$. They have shown that better agreement with the measured values of magnetic moments is obtained than by the Schmidt diagram.

Energy calculation indicated that the lowest states of odd nuclei in the configuration j^n are expected to be those of smallest isotopic spin, resultant angular momentum $J=j$ and lowest seniority number, which arises from short range attractive forces[4,5].

We shall treat magnetic moments on the same assumption as aforementioned, and making use of fractional-parentage-coefficients (to which we shall refer simply as *f.p.c*) introduced by Flowers and Edmonds[5] for *jj* coupling, the general formula of magnetic moments for odd nuclei, in the lowest states, will be obtained.

I. THE VALUE OF THE MAGNETIC MOMENT FOR THE LOWEST STATE

The operator of the z component of the magnetic moment in *jj* coupling for n equivalent particles is

$$\mu_z = (1/2j)\left\{(\mu_n+\mu_p)J_z+(\mu_n-\mu_p)\sum_i 2t_{\zeta_i}j_{z_i}\right\} \qquad (1)$$

where μ_p and μ_n are the Schmidt values of a proton and neutron appropriate to the given values of l and j, t_ζ is the ζ component of the isotopic spin of a particle (i.e. $t_\zeta=-1/2$ for a proton, $t_\zeta=1/2$ for a neutron).

According to the vector model (compare the anomalous Zeeman effect), the following is obtained for the effective magnetic moment μ_z:

$$\mu_z = (J_z/2j)\left\{(\mu_n+\mu_p)+(\mu_n-\mu_p)\,[2T_\zeta/T(T+1)j(j+1)]\sum_i (t_i\,T)\,(j_i J)\right\} \qquad (2)$$

The magnetic moment μ which is the maximum value of μ_z i.e. its value in the substate $J_z=j$ is given by :

$$\mu = \beta\mu_n+(1-\beta)\mu_p \qquad (3)$$

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where β is given by the equation

$$\beta = (1/2)\left\{1+[2T_\zeta/T(T+1)j(j+1)]\sum_i (t_i T)(j_i J)\right\} \tag{4}$$

and for a conjugate nucleus (a conjugate nucleus of a given nucleus is one in which the number of protons is equal to the number of neutrons of the given nucleus and vice-versa)

$\mu_{conj} = (1-\beta)\mu_n + \beta\mu_p$, where β is the same as that in (3), namely has the same value for T_ζ. (3¹)

Thus, for conjugate configuration one has to change the coefficients of μ_p and μ_n. Making use of *f.p.c.*, and owing to the antisymmetry of the wave functions[6], we obtain for β the following expression:

$$\beta = (1/2)\left\{1+[2T_\zeta n/T(T+1)j(j+1)]\sum_{\substack{\gamma s_1 t_1\\ T_1 J_1}} \langle T_1 J_1, t_n j_n, TJ \mid (t_n \cdot T)(j_n J) \mid T_1 J_1, t_n j_n, TJ\rangle \cdot \langle j^{n-1}(\gamma, s_1 t_1, T_1 J_1) \mid \} j^n, st, Tj\rangle^2\right\} \tag{5}$$

We are interested in the case $s=1$, $t=1/2$, and for this case, using the reciprocity equation given by Flowers and Edmonds[4] (1952 Eq. (9)) which is analogous to that of Racah[7] we have

$$\langle j^{n-1}(\gamma, s_1 t_1, T_1 J_1) \mid \} j^n, 1, 1/2, Tj\rangle^2 = A(nT, \gamma s_1 t_1 T_1)\cdot(2J_1+1)$$

and let us put $\sum_\gamma A(nT,\gamma s_1 t_1 T_1) = B(nT, s_1 t_1 T_1)$.

Substituting this and the expectations value of $(t_n \cdot T)(j_n J)$ into (5) we get

$$\beta = (1/2)\left\{1+[T_\zeta n/2T(T+1)j(j+1)]\sum_{s_1 t_1 T_1 J_1} B(nT,s_1 t_1 T_1)(2J_1+1)[2j(j+1) - J_1(J_1+1)][T(T+1)+3/4-T_1(T_1+1)]\right\} \tag{6}$$

From (6) we see that the problem is to calculate the coefficients $B(nT,s_1 t_1 T_1)$.

II. Calculation of the coefficients $B(nT,s_1 t_1 T_1)$.

For a given n we have six coefficients $B(nT,s_1 t_1 T_1)$, since T_1 may have only two values: $T \pm 1/2$ and s_1 only two values: 0 and 1. Furthermore, for $s_1=0$, t_1 may have only the value 0, and for $s_1=2$, t_1 may have only the values 0 and 1.

But since in the configuration j^{n-1}, states corresponding to $(s_1,t_1)=(0,0)$ exist only when $1/2(n-1)-T_1$ is even, and those corresponding to $(s_1,t_1)=(2,0)$ exist only when $1/2(n-1)-T_1$ is odd (see Appendix), two coefficients, $B(nT,s_1 t_1 T_1)$ vanish identically, namely

$$\left.\begin{aligned} B(nT,00T+1/2) &= 0\\ B(nT,20T-1/2) &= 0\end{aligned}\right\} \quad \text{for} \quad N_o > N_e \tag{7}$$

$$\left.\begin{aligned} B(nT,00T-1/2) &= 0\\ B(nT,20T+1/2) &= 0\end{aligned}\right\} \quad \text{for} \quad N_e > N_o \tag{8}$$

where N_e and N_o are the even and odd number of N or P (N—number of neutrons, P — number of protons) outside the core, i. e. $N_e+N_o=n$. Then for calculating the remaining coefficients we need four relations. Using Eq. (23) of Racah[6] we get three relations, which follow from the normalization of the wave function and from the equations $\sum_i t_{\zeta_i} = T_\zeta$, $\sum_i j_{z_i} = j_z$.

$$\left.\begin{aligned}
&B(nT,00T+{}^1/_2)+{}^1/_2(2j-1)(2j+2)B(nT,21T+{}^1/_2)+{}^1/_2(2j+1)(2j+2)B(nT,20T+{}^1/_2)\\
&\quad=(n-2T)(T+1)/n(2T+1);\\
&B(nT,00T-{}^1/_2)+{}^1/_2(2j-1)(2j+2)B(nT,21T-{}^1/_2)+{}^1/_2(2j+1)(2j+2)B(nT,20T-{}^1/_2)\\
&\quad=(n+2T+2)\cdot T/n(2T+1);\\
&B(nT,00T+{}^1/_2)+{}^1/_2(2j-1)B(nT,21T+{}^1/_2)-{}^1/_2(2j+1)B(nT,20T+{}^1/_2)+\\
&\quad+B(nT,00T-{}^1/_2)+{}^1/_2(2j-1)B(nT,21T-{}^1/_2)-{}^1/_2(2j+1)B(nT,20T-{}^1/_2)=1/n.
\end{aligned}\right\}\quad(9)$$

Another relation follows from the reciprocity equation

$$\langle j^{4j+2-n}(1,{}^1/_2,Tj)\,|\,\}j^{4j+3-n}(00,T_1J_1=0)\rangle^2=[n(2T+1)(2j+1)/(4j+3-n)(2T_1+1)]\cdot\langle j^{n-1}(00,T_1J_1=0)\,|\,\}j^n(1,{}^1/_2,Tj)\rangle^2$$

which is a particular case of the analogous reciprocity equation of Racah[6] (Eq. 19). The left side of the equation may be calculated from the normalization of the wave function and the value $T_1\zeta$ of the configuration j^{4j+3-n}, and then we obtain

$$\left.\begin{aligned}
B(nT,00T-{}^1/_2)&=(4j+4-n-2T)/2n(2j+1)\quad\text{for } N_o>N_e\\
B(nT,00T+{}^1/_2)&=(4j+6-n+2T)/2n(2j+1)\quad\text{for } N_e>N_o
\end{aligned}\right\}\quad(10)$$

The coefficients $B(nT,s_1t_1T_1)$ can be derived from (9) and (10), and substituting them into (6) we obtain for the β of (3) which corresponds to a nucleus containing an even number of neutrons.

$$\left.\begin{aligned}
\beta&=N_e/(2j+2)(2T+2)\quad\text{for } N_o>N_e\\
\beta&=(2j+1-N_e)/(2j+2)(2T+2)\quad\text{for } N_e>N_o
\end{aligned}\right\}\quad(11)$$

According to (3) (3[1]), we obtain for the magnetic moments the formula

$$\mu=\beta\mu_e+(1-\beta)\mu_o\quad(12)$$

where μ_e and μ_o are the Schmidt values of the even and odd particle, and β is given by (11).

III. DISCUSSION

From the formula of magnetic moments derived in section 2, we can see that in the case of like particles, namely a configuration of protons only or neutrons only, the magnetic moments have the Schmidt values, although generally they should deviate and fall between the Schmidt lines. The deviation for conjugate nuclei should be of the same value but of opposite direction. Measured magnetic moments indicate that the law of opposite directions exists although the deviations are not equal. Furthermore, all measured magnetic moments of odd nuclei except H^3, He^3, C^{13}, N^{15} fall between the

Schmidt lines. For some cases a better agreement with the measured moments is obtained by using our formula than by the Schmidt diagram, but even our formula does not fit satisfactorily.

APPENDIX

The wave functions $\psi(j^m[\lambda](\sigma)JJ_2TT_2)$ are characterised by the Young tableaux $[\lambda]$ (σ) which correspond to the irreducible representations of the unitary group $U(2j+1)$ and the symplectic group $S_p(2j+1)$ respectively. There is a one to one correspondence between $[\lambda]$ and the isotopic spin, (σ) and the number pair (s,t), the length of the two columns of $[\lambda]$ is given by $^1/_2m\pm T$, that of (σ) by $^1/_2s\pm t$.[2] We shall denote by $\widetilde{P}(a,b)$ and $\widetilde{P}_o(a,b)$ the irreducible representations of $U(2j+1)$ and $S_p(2j+1)$ respectively, where a and b measure the length of the columns of a scheme, $a \geqq b$.

Under limitation of $U(2j+1)$ to its symplectic subgroup $S_p(2j+1)$, $\widetilde{P}(a,b)$ is generally reducible, and the reduction law is given by Littlewood[8] (p. 295), in particular

$$\widetilde{P}(a,0) = \sum_{x=0}^{[a/2]} \widetilde{P}_o(a-2x,0) \tag{1}$$

and

$$\widetilde{P}(a,1) = \sum_{x=0}^{[a/2]-1} \widetilde{P}_o(a-1-2x,0) + \sum_{x=0}^{[(a-1)/2]} \widetilde{P}_o(a-2x,1) \tag{2}$$

where $[f]$ denotes the greatest integer not greater than f; it follows from (1) that $\widetilde{P}_o(a,o)$ may be written as

$$\widetilde{P}_o(a,0) = \widetilde{P}(a,0) - \widetilde{P}(a-2,0) \tag{3}$$

On the other hand, from Littlewood[8] (p. 94), we get

$$\widetilde{P}(a,0) \times \widetilde{P}(1,1) = \widetilde{P}(a+1,1) \tag{4}$$

and

$$\widetilde{P}(a,0) \times \widetilde{P}(b,0) = \sum_{x=0}^{b} \widetilde{P}(a+b-x,x) \tag{5}$$

therefore $\widetilde{P}(a,b)$ can be written as

$$\widetilde{P}(a,b) = \widetilde{P}(a,0) \times \widetilde{P}(b,0) - \widetilde{P}(a+1,0) \times \widetilde{P}(b-1,0) \tag{6}$$

From the reality of the characters of $S_p(2j+1)$ it follows that:

(i) $\widetilde{P}_o(0,0)$ appears in the reduction of the Kronecker product $\widetilde{P}_o(k,0) \times \widetilde{P}_o(l,0)$ only when $k=l$.

(ii) $\widetilde{P}_o(1,1)$ appears in the reduction of $\widetilde{P}_o(k,0) \times \widetilde{P}_o(l,0)$ the number of times $\widetilde{P}_o(k,0)$ appears in the reduction of $\widetilde{P}_o(l,0) \times \widetilde{P}_o(1,1)$.

According to (2), (3) and (4) we have

$$\begin{aligned} \widetilde{P}_o(l,0) \times \widetilde{P}_o(1,1) &= [\widetilde{P}(l,0) - \widetilde{P}(l-2,0)] \times \widetilde{P}(1,1) \\ &= \widetilde{P}(l+1,1) - \widetilde{P}(l-1,1) \\ &= \widetilde{P}_o(l,0) + \widetilde{P}_o(l+1,1) \end{aligned}$$

Therefore $\widetilde{P_0}(1,1)$ appears in $\widetilde{P_0}(k,0)\times\widetilde{P_0}(l,0)$ only when $k = l > 0$. From (6), (1), (i), (ii), we obtain that $\widetilde{P_0}(0,0)$ appears in the reduction of $\widetilde{P}(a,b)$, $([b/2]-[(b-1)/2])$ times, and $\widetilde{P_0}(1,1)$, $([(b+1)/2]-[b/2])$ times. Then since $\widetilde{P_0}(0,0)$ corresponds to $(s,t) = (0,0)$, $\widetilde{P_0}(1,1)$ corresponds to $(s,t) = (2,0)$ and b is $\frac{1}{2}m-T$, therefore $(s,t)=(0,0)$ appears only when $\frac{1}{2}m-T$ is even, and $(s,t) = (2,0)$ only when $\frac{1}{2}m-T$ is odd.

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PRODUCTION OF ANTIPROTONS IN *P-P* COLLISIONS

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If one assumes that the proton wave function obeys Dirac's equation, one should expect that anti-protons exist. This possibility has become more interesting in recent years, with the planning and construction of accelerating machines in the Bev range. If a proton pair is to be produced by the collision of two protons, then, in the centre of mass system, each of the initial particles must have a total energy of at least $2M$, where M is the rest mass of the proton. By transforming to the rest frame of one of the particles, it is easily shown that at threshold the incident proton has a total energy of $7M$, or a kinetic energy of $6M = 5.6$ Bev. It is expected that such energies will be attained in the near future.

In this paper, the cross-section for pair production by the above process is calculated, using the Feynman method[1] and assuming pseudoscalar mesons with pseudovector coupling. The calculations were carried out only in the neighbourhood of threshold, since much higher energies will probably not be experimentally attainable in the near future. The matrix elements are expanded in a power series in the components of the 3-momenta of the final particles in the centre-of-mass system, and only the zero and first degree terms are considered.

Some information about the angular distribution and the energy dependence may be obtained without performing the algebraic calculations. For example, it is shown that the total cross section near threshold varies as the 9/2 power of the total kinetic energy; this conclusion holds for any of the usual weak coupling theories. In contrast, a 7/2 power dependence is obtained by Fermi[2] in his statistical theory of high energy processes.

FEYNMAN DIAGRAMS

A typical Feynman diagram for the process under consideration is shown in Figure 12 where the solid lines represent protons and the dashed lines mesons. Particles 1 and 2 are the initial protons, 4, 5, and 6 are the final protons, and 3 is the antiproton. The direction of the arrow associated with 3 is consistent with Feynman's description of an antiparticle as one whose phase increases in the negative time direction[2]. It is seen that Figure 1 represents a fourth order process.

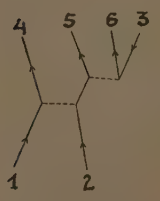

Figure 1

Figure 2

Figure 1 may be defined toplogically as consisting of three proton lines, each having a defined sense of phase increase, but with the direction of time increase not defined, and of two connecting meson lines. With this definition, it is seen that Figure 1 is topologically equivalent to a diagram for the

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mutual scattering of three protons, as well as to the schematic diagram of Figure 2. Furthermore, there is no fourth order diagram containing three proton lines which is topologically different from Figure 2.

There are 36 fourth order diagrams for the process, obtained from Figure 1 by the six permutations of particles 1, 2, and 3 times the six permutations of 4, 5 and 6. Only such changes must be made in the diagram after each permutation as are necessary to maintain the correct time direction of phase increase for each particle, with the anti-proton always labelled as particle 3. From a consideration of Figure 2, one can show that these are the only possible fourth order diagrams.

NOTATION

P_i^{λ} = 4-momentum of the ith particle (i=1, 2, . . . 6)

γ_λ = Dirac matrix (λ = 1, 2, 3, 4)

$\gamma_5 = \gamma_1 \gamma_2 \gamma_3 \gamma_4$

$\mathbf{P}_i = P_i^{\lambda} \gamma_\lambda = P_i^4 \gamma_4 - P_i^1 \gamma_1 - P_i^2 \gamma_2 - P_i^3 \gamma_3$

u_i = Dirac state of ith particle, in the momentum representation.

E = energy of either initial proton in the centre of mass system.

M = proton mass.

μ = meson mass.

ε = $E - 2M$, so that 2ε is the sum of the kinetic energies of the four emitted particles.

g = mesic charge of a proton.

Natural units ($h/2\pi = c = 1$) are used.

MATRIX ELEMENTS

The matrix elements for the transition are obtained from the diagrams by application of the rules given by Feynman[1]. For pseudoscalar mesons with pseudovector coupling, the matrix element term corresponding to Figure 1 is:

$$\left(\frac{g^2}{\pi i}\right)^2 \frac{[\bar{u}_4 \gamma_5 (\mathbf{P}_4 - \mathbf{P}_1) u_1][\bar{u}_6 \gamma_5 (\mathbf{P}_6 - \mathbf{P}_3) u_3][\bar{u}_5 \gamma_5 (\mathbf{P}_6 - \mathbf{P}_3)(\mathbf{P}_1 + \mathbf{P}_2 - \mathbf{P}_4 + M)\gamma_5 (\mathbf{P}_4 - \mathbf{P}_1) u_2]}{\mu^4 [(P_1 + P_2 - P_4)^2 - M^2][(P_4 - P_1)^2 - \mu^2][(P_6 - P_3)^2 - \mu^2]} \tag{1}$$

Each matrix element consists of the antisymmetric sum of 36 terms, one for each Feynman diagram. There is one matrix element for each of the 64 spin states that the six particles can have.

The transition probability is given by the well-known formula

$$W = 2\pi \rho(\vec{p}_3, \vec{p}_4, \vec{p}_5, \vec{p}_6) 1/4 \sum_{\alpha} |M_\alpha|^2 \tag{2}$$

where ρ is the density of final states, M_α is the matrix element corresponding to a given assignment of spins to the six particles, and the sum is taken over all spin states. The factor of 1/4 arises from the fact that for the initial protons an average, rather than a sum, is taken over the spin states.

The process of summing and averaging over spin states is usually carried out by using the spur method[3]. In simpler problems, this method involves less labour than does the direct calculation of the matrix element for each assignment of spins. However, the

large number of diagrams in the present problem makes even the spur method complex. On the other hand, it turns out that if each matrix element is calculated directly, certain arguments may be applied which greatly simplify the threshold calculation.

Furthermore, one may use these arguments, without performing the calculations, to obtain some information about the angular and energy dependence of the process. The antisymmetry of the matrix element with respect to interchange of two final protons reduces, in the case of two particles with the same spin, to antisymmetry with respect to interchange of the corresponding momenta. Since at least two of the three final protons must have the same spin, no matrix element will contain terms which are completely symmetric with respect to the three corresponding momenta.

Examination of expression (1) shows that if a matrix element is expanded in powers of the components of 3-momenta of the four emitted particles, only positive powers will appear. From the preceding argument, it follows that the constant term must be zero in such an expansion, independent of the assignment of spins. The first degree terms, which may be present for some spin assignments, must be independent of the momentum components of the antiproton, since a term containing the antiproton momentum would be symmetrical in the momenta of the protons.

The leading terms of a similar expansion of the differential cross-section will therefore be independent of the antiproton momentum and will be a homogeneous quadratic function of the momentum components of the three final protons.

The total cross-section is obtained by integrating equation (2) over the momentum space of the four emitted particles and dividing by the incident flux. The density of states, p, contains the energy and momentum δ-functions which restrict the available region of momentum space to that allowed by energy and momentum conservation.

The integration over the 12 momentum coordinates was performed by transforming to a system consisting of a radial coordinate, R, proportional to the square root of the total emitted kinetic energy, and 11 angular coordinates, so chosen as to make the limits of each integration independent of the remaining coordinates. The integrations over the angular coordinates yield numerical factors. The integration over R contains the energy δ-function and therefore substitutes a constant times $\varepsilon^{1/2}$ for R.

It is easily shown that the available volume in momentum space (taking into account the restrictions imposed by the δ-functions) is proportional to R^7. The sum of squares of matrix elements is quadratic in the momentum components, and since each component is proportional to R in the transformation, the integral is proportional to R^9, or $\varepsilon^{9/2}$. Since we are dealing with a fourth order process, the constant of proportionality must include a factor of g^8. The total cross section is therefore

$$\sigma = K(g^2/\pi i)^4 \ (\varepsilon/M)^{9/2} \qquad (3)$$

The arguments used to obtain this equation are valid for all the usual weak coupling theories, so that, in general, one can expect that only the value of K will differ from one theory to another in this approximation.

The symmetry arguments discussed above may be extended to reduce the labor of calculation. For example, it may be shown that of the 64 matrix elements (each corresponding to a given assignment of spins) only two need be calculated in detail. Of the remaining ones, some do not contribute near threshold, because their leading terms

must be of higher degree; others may be obtained from the computed ones by simple permutations of indices in the results. Similarly, within each matrix element, not all the terms corresponding to the various Feynman diagrams need be calculated in detail.

RESULTS AND DISCUSSION

Calculations were made only for the case of pseudoscalar mesons with pseudovector coupling. The total cross-section near threshold is given by equation (3), with $K \quad 1.2 \times 10^{-25}$ cm^2. Since the matrix element is independent of the antiproton momentum, this particle will be emitted spherically symmetrically.

The above cross-section may be compared with that obtained by Fermi[2] in his statistical theory of high energy nuclear processes: $\sigma \sim 7 \times 10^{-26} \varepsilon^{7/2}$. Fermi assumes that the matrix element is independent of ε, so that only the volume element in momentum space contributes to the energy dependence. It is seen from the preceding section that such an assumption leads to a 7/2 power dependence on ε. Fermi's theory is a strong coupling theory while the present calculations are based on a weak coupling theory; to a certain extent, the two results represent opposite limiting cases.

Taketani and Machida[4] have calculated the cross-section for the same process, using the pseudoscalar-pseudoscalar theory. They obtained a fourth power dependence on ε, which disagrees with the conclusions of the preceding section. In their discussion, they omit some of the Feynman diagrams in order to simplify a comparison with another problem in the same paper. It is not clear, however, whether the corresponding contributions are also omitted from their calculations. In any case, their result is strange, since such an omission should still lead to a half-odd-integer power dependence on ε. The integral power may be the result of an approximation which they make in calculating the volume element in momentum space, but which they do not describe in detail.

A complete discussion of the results of the present paper must await experimental investigation.

The author wishes to express his gratitude to Professor Robert Serber for suggesting the problem and for his helpful discussions.

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ON THE SECOND HARMONIC OF THE ANNUAL VARIATION OF SOLAR RADIATION AND AN APPLICATION OF ITS ZEROS

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I. INTRODUCTION

In the past, most applications of the annual variation of solar radiation reaching the outer atmosphere have limited their attention to the first harmonic, i.e. the annual wave. This neglect of the semi-annual wave is somewhat unjustified as it is known that the annual wave vanishes at the equator (more precisely, at latitude 3°25′N) and that in the equatorial belt, the annual variation of solar radiation is a double wave. It is the purpose of the present note to examine the latitudinal variation of the semi-annual wave and point out a significant application of that latitudinal variation. It should be emphasized that the solar radiation considered is the one reaching the outer atmosphere.

Milankovitch[1] has given the theoretical form of the harmonic coefficients of the annual variation of solar radiation as a function of latitude. It is shown by his results that the coefficients are doubly infinite series in a trigonometric function of the latitude φ. Thus, the coefficients are of a rather complicated structure despite a simplifying assumption involved in their derivation, viz., that the distance between the Sun and the Earth is constant. To obtain a more tractable expression, we shall adopt an additional assumption as well as terminate the (infinite) series involved after the first few terms. Although the coefficient so obtained is of a less general character than that given by Milankovitch, it should be quite satisfactory for most geophysical applications.

II. THE SECOND HARMONIC

The daily amount of solar radiation Q_s reaching the Earth in latitude φ at the outer atmosphere is given, to a close approximation, by the following expression (ref. 2, p. 88):

$$Q_s = (2I/\Omega) \sin\varphi \sin\delta (H - \operatorname{tg} H), \qquad (1)$$

where I is the solar "constant", Ω the angular speed of rotation of the Earth about its axis, δ the solar declination and H the hour angle between sunrise and noon (= hour angle between noon and sunset). Expressed in this manner, Q_s is in gcal/cm²/day. Excluding regions of the Earth poleward of the polar circles, H, φ, and δ are linked by the equation

$$\cos H = -\operatorname{tg}\varphi \operatorname{tg}\delta. \qquad (2)$$

H is usually referred to as "half-day length".

In view of Eq. (2), (1) becomes, apart from the constant factor $2I/\Omega$,

$$\sin\varphi \sin\delta (H - \operatorname{tg} H) = \sin\varphi \sin\delta [\arccos(-\operatorname{tg}\varphi \operatorname{tg}\delta)] + \cos\varphi \cos\delta (1 - \operatorname{tg}^2\varphi \operatorname{tg}^2\delta)^{1/2}. \qquad (3)$$

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The term containing the radicand may be rewritten:

$$(\cos^2\varphi\cos^2\delta-\sin^2\varphi\sin^2\delta)^{1/2} = \cos\varphi\,[1-(\sin^2\delta/\cos^2\varphi)]^{1/2}. \quad (4)$$

To an approximation, $\sin\delta = \sin\iota\sin t = a\sin t$, where $a = \sin\iota$, $\iota = 23°27'$ being the angle of inclination of the ecliptic; $a = 0.398$, very nearly. For $-[(\pi/2)-\iota] < \varphi < [(\pi/2)-\iota] = 66°33'$, $(\sin\iota/\cos\varphi) < 1$ and under this condition, (4) may be expanded as follows:

$$\cos\varphi\,[1-(a^2/\cos^2\varphi)\sin^2 t]^{1/2} = \cos\varphi\,[1-{}^1/_2(a^2/\cos^2\varphi)\sin^2 t-{}^1/_8(a^4/\cos^4\varphi)\sin^4 t + \varepsilon(\varphi,t)]. \quad (5)$$

The residual term $\varepsilon(\varphi,t)$ is of the sixth order in $\sin t$ and will be neglected.

For the term $\arccos(-\operatorname{tg}\varphi\operatorname{tg}\delta)$ in (3), the expansion

$$\arccos(-\operatorname{tg}\varphi\operatorname{tg}\delta) = (\pi/2) + \operatorname{tg}\varphi\operatorname{tg}\delta + ({}^1/_2\cdot{}^1/_3)(\operatorname{tg}\varphi\operatorname{tg}\delta)^3 + \ldots \quad (6)$$

will be substituted with the terms of fifth order and higher omitted. For $\operatorname{tg}\delta$ $(-\iota \leqslant \delta \leqslant +\iota)$ we shall write $b\sin t$, $b = \operatorname{tg}\iota$, $\iota = 23°27'$; $b = 0.434$, very nearly. With this, and in view of Eq.'s (4) to (6), (3) becomes after some re-arrangement:

$$\sin\varphi\sin\delta\,(H-\operatorname{tg}H) = \cos\varphi + (\pi/2)\,a\sin\varphi\sin t + (a/\cos\varphi)\,(b\sin^2\varphi - a/2)\sin^2 t + (a/\cos\varphi)\,[(b^3/6)(\sin^4\varphi/\cos^2\varphi) - (a^3/8)(1/\cos^2\varphi)]\sin^4 t. \quad (7)$$

If (7) is multiplied by $(1/\pi)\sin 2t$, and integrated with respect to t between $t = 0$ and $t = 2\pi$, it is seen that all terms vanish. Hence, q_2, the second Fourier coefficient of the sine series, vanishes. If (7) is multiplied by $(1/\pi)\cos 2t$ and integrated as necessary, it is found that the first and second terms on the right hand side vanish. In the third term,

$$(1/\pi)\int_0^{2\pi}\sin^2 t\cos 2t\,dt = -{}^1/_2; \quad (8)$$

in the fourth term, similarly,

$$(1/\pi)\int_0^{2\pi}\sin^4 t\cos 2t\,dt = -{}^1/_2. \quad (9)$$

Substituting (8) and (9) into the integral of (7), it is found that p_2, the second coefficient of the cosine series:

$$p_2 = -(a/2\cos\varphi)\,[b\sin^2\varphi + (b^3/6)(\sin^4\varphi/\cos^2\varphi) - a/2 - (a^3/8)(1/\cos^2\varphi)]. \quad (10)$$

This is the equation we have intended to derive.

It follows from differentiation of (10) with respect to φ, that in the interval $-66°33' < \varphi < +66°33'$, p_2 reaches a maximum value at the equator. In the same interval, p_2 will vanish for

$$\sin^2\varphi = \frac{-3(2b+a)+\left]9(2b+a)^2+3ab(b^2-6)(a^2+4)\right]^{1/2}}{2b(b^2-6)}, \quad (11)$$

which yields the real zeros of p_2. As $a = 0.398$ and $b = 0.434$, $\sin^2\varphi = 0.481$, $\varphi = \pm 44°$, approximately, or *the semi-annual wave vanishes along a latitude circle in the middle latitudes of each of the two hemispheres*. Hence the wave is symmetrical about the equator. $p_2 > 0$ for $-44° < \varphi < +44°$, vanishes at $\varphi = \pm 44°$, and is negative beyond. It is possible to show that p_2 continues to be negative as far as the poles with its absolute values increasing with increasing latitude.

The equations $\sin \delta = a \sin t$ and $\operatorname{tg} \delta = b \sin t$ imply that $t = 0$ at the spring equinox when the Sun's declination angle vanishes in the course of passing from negative to positive values. Counting the time from the spring equinox, we find that the phase angle A_2 of the semi-annual wave is 90° for the region equatorward of the middle latitudes ($p_2 > 0$, $q_2 = 0$) and 270° poleward of the middle latitudes ($p_2 < 0$, $q_2 = 0$). If $t = 0$ at mid-January, it follows that A_2 is 320° equatorward, and 140° poleward of the middle latitudes. These conclusions are in close agreement with results of a numerical harmonic analysis in Table I based on data which do not involve the simplifying assumptions used in obtaining the form of p_2 (Eq. (10)). It is also seen from the Table that the latitudes of both the equatorial maximum and that of the zeros of p_2 are shifted slightly toward the North Pole compared with the results of the somewhat simplified theoretical analysis.

TABLE I

Harmonic constants of the semi-annual wave of solar radiation, for the northern hemisphere (Prepared from Angot's data, published in Shaw[3])

Notes: 1) Time is counted from mid-January; 2) a_2 is the amplitude and A_2 the phase angle of the wave of form $a_2 \sin (2t + A_2)$; 3) p_2 and q_2 are coefficients of the cosine and sine series, respectively; 4) a_2, p_2 and q_2 are in gcal/cm²; 5) x — value is less than 1 gcal/cm².

Latitude, °N	a_2	p_2	q_2	A_2
0	34.4	— 22.5	+ 26.0	319
10	35.2	— 23.3	+ 26.4	319
20	32.6	— 21.5	+ 24.5	319
30	26.4	— 17.7	+ 19.6	318
40	15.7	— 10.8	+ 11.4	317
50	x	x	x	—
60	27.7	+ 17.9	— 21.1	140
70	89.5	+ 58.4	— 67.8	139
80	174.7	+114.7	—131.7	139
90	209.0	+136.2	—158.6	139

III. Application

While the maximum value of p_2 at the equator is as expected, the fact of vanishing of this wave for the middle latitudes is somewhat surprising. These zeros of p_2 appear to be relevant to the problem of annual variation of energy transfer to the atmosphere from the oceans, and other extensive water bodies in the middle latitudes.

It has been found[4] that in the middle latitudes, the annual variation of energy (latent and sensible heat) transfer from the oceans to the atmosphere is a double wave. More recent studies[5] indicate that this is also true for lakes of some depth. No theoretical investigation of a general nature is available in this connection and it would appear that such an investigation would meet a number of difficulties. However, an explanation of the double wave character of the annual variation may be obtained from a consideration of Fourier waves.

The energy transfer from a water body to the atmosphere normally may be represented by the equation

$$Q_a = Q_r - Q_0, \tag{12}$$

where Q_a is the sum of latent and sensible heats transferred from the water to the air, Q_r the net radiation surplus at the water surface and Q_0 the change in heat content of the water body. The latter change is connected with the surface-temperature wave

of the water. It is shown elsewhere[6] that processes of heat conduction lead to an increase in importance of the amplitude of the second harmonic wave, relative to the importance of the amplitude of the second harmonic of the surface-temperature wave. In the present case, the second harmonic is the semi-annual wave. Thus processes of heat conduction in the water result in an annual variation of heat content of the water body, for which the semi-annual wave is prominent.

It is reasonable to assume for a first approximation that the annual variation of Q_r, the net radiation surplus at the water surface, is parallel to the annual variation of Q_s, the solar radiation reaching the outer atmosphere. This assumption implies that the semi-annual wave of Q_r is negligible for the middle latitudes. Now, detailed analysis shows that the annual waves of Q_r and Q_θ are nearly in phase, the phase difference amounting to about one month, so that the difference of the two waves (Eq. (12)) is small. Because the semi-annual wave of Q_s, and by assumption that of Q_r too, are negligible for the middle latitudes, the difference $Q_r - Q_\theta$ is effectively controlled by the semi-annual wave of Q_θ which is relatively strong. It thus follows that the annual variation of Q_a, the energy transfer from the water surface to the atmosphere, is a double wave.

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AN ATTEMPT TO DETECT THE ELECTROLYTIC MIGRATION OF RADIOACTIVE INTERSTITIAL SILVER IONS PRODUCED IN A SILVER CHLORIDE CRYSTAL BY SLOW NEUTRON IRRADIATION

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INTRODUCTION

The phenomena of diffusion and ionic conduction in ionic crystals are ascribed to the presence of mobile lattice defects[1]. These may be of two types: a) Frenkel defects, when an ion occupies an interstitial position, leaving a vacant lattice point or "hole", b) Schottky defects, when an ion is displaced to the surface of the crystal, leaving a hole in the body of the crystal. In silver chloride and silver bromide crystals, it appears that the predominant lattice defect below 300°C is of the Frenkel type, due to the interstitial silver ions[2]. In the case of Frenkel defects, the number of defects in equilibrium at an absolute temperature T is proportional to $e^{-W/2kT}$, where W is the energy required to move an ion from its normal position in the lattice to an interstitial position. The mobility of a Frenkel defect in an applied electric field is proportional to $e^{-U/kT}$, where U is the activation energy for diffusion, that is to say, the height of the potential barrier which an interstitial ion must surmount in passing to a neighbouring interstitial position. The electrical conductivity arising from the migration of Frenkel defects in an electric field is equal to the product of the concentration of the defects and their mobility, and therefore should be proportional to $e^{-(U + W/2)kT}$.

A measurement of the dependence of conductivity on temperature is clearly insufficient to determine the activation energies W and U separately. In order to determine W and U, recourse has been made to rather special and indirect methods. For example, Koch and Wagner[3] measured the conductivity of silver chloride and silver bromide crystals which contained small quantities of cadmium chloride. Each divalent cadmium ion displaces two monovalent silver ions, leaving a number of holes equal to the known number of cadmium atoms. At sufficiently low temperatures, the conductivity is determined by this constant number of holes, and U the activation energy for diffusion can be determined directly from the dependence of conductivity on temperature. Koch and Wagner assumed the mobility of an interstitial ion is equal to that of a hole. For AgCl, Koch and Wagner obtained for W and U the values of 1.1 e.v. and 0.25 e.v. respectively, and for AgBr, the corresponding values of 0.87 e.v. and 0.35 e.v. Seitz[2] reports, however, on the basis of later experiments, a value of 0.11 e.v. for U in the case of AgBr. This would lead one to suspect that U for AgCl might also be much lower than the value of 0.25 obtained by Koch and Wagner.

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THE POSSIBILITY OF PRODUCING RADIOACTIVE INTERSTITIAL IONS BY SLOW NEUTRON BOMBARDMENT

Under certain circumstances one might produce radioactive interstitial ions by a process analogous to that occurring in the Szilard-Chalmers effect. In the latter effect, it is well known that the nuclear recoil arising from the emission of energetic γ-rays after slow neutron capture is sufficient to break the chemical bonds binding the reacting atom, thus releasing the atom which is often radioactive. Analogously, one might expect that when an ionic crystal, containing nuclei which capture slow neutrons, is irradiated with slow neutrons, a large fraction of the radioactive ions produced will be displaced by the nuclear recoil to interstitial positions. Under certain conditions one might then be able to observe the mobility of these radioactive ions in an applied electric field, by radioactive tracer techniques. Thus if the crystal is placed between two electrodes, an application of a sufficiently large D.C. field should lead to a concentration of radioactive ions near the electrode of opposite polarity to that of the interstitial ions. An experiment designed to detect such a change in concentration in the case of AgCl crystals is described below.

Necessary conditions for the detection of the migration of radioactive interstitial ions are:

1) The concentration of holes must be sufficiently low, in order to minimize the probability of recombination of a radioactive interstitial ion with a hole.

2) The migration time in the field must not be too long compared with the radioactive decay time; otherwise the decay of the ions in passage will reduce the probability of detection.

3) There should be no direct exchange between interstitial ions and normally situated neighbours. There seems to be no evidence that this process occurs and, à priori, the probability may be expected to be exceedingly small.

DESCRIPTION OF EXPERIMENTS

The experiments were performed on single crystals of silver chloride. The two stable isotopes of silver, Ag^{107} and Ag^{109}, both have favourable nuclear properties. Both isotopes capture thermal neutrons with a high cross-section (44×10^{-24} cm^2 and 97×10^{-24} cm^2 respectively) leading to Ag^{108}, a radioactive isotope with a half life of 2.3 minutes, and to Ag^{110} with a half life of 24 seconds. The maximum energies of the β-particles are 2.85 mev and 2.8 mev respectively. Since only a weak neutron source was available (80 millicuries polonium-beryllium), high cross-sections for neutron capture were essential.

Silver chloride crystals (supplied by the Harshaw Chemical Company, U.S.A.) of superficial dimensions 12×13 mm and thickness 3.2 mm were used. The crystal was placed between platinum electrodes, between which a D.C. voltage of about 4000 volts could be applied. The crystal and electrodes were enclosed in a thin-walled aluminium box which was placed in the centre of a large block of paraffin wax ($30\times30\times30$cm^3). The polonium-beryllium neutron source could be placed near the box, yielding a thermal neutron flux of about 100/cm^2 sec at the surface of the crystal (as measured with thin silver and indium detectors). The space between the aluminium box and the paraffin moderator could be packed with solid CO_2 in order to reduce the temperature of the

crystal to −80°C. Most of the measurements were carried out at this temperature. At room temperature the equilibrium concentration of holes is about 10^{12}—10^{13} per cc[2] and the probability of recombination would be too large. Moreover, at room temperatures application of a large field leads to the production of threads of colloidal silver inside the crystal, which is rapidly spoilt. At very low temperatures, on the other hand, the migration time would be too long compared to the radioactive decay time. As a compromise between these extremes it was decided to work at dry ice temperatures.

Neutron irradiations of about ten minutes duration were carried out (sufficient to yield approximately the saturation intensity of radioactive silver), at first without an applied field. After irradiation the crystal was quickly withdrawn and placed, under conditions of standard geometry, immediately under the thin window (1.5 mg/cm[2]) of an end-on Geiger counter. The Geiger counter was screened with lead and the background counting rate was 15 counts per minute. After irradiation the crystal showed an activity of about 50 counts above background during the first minute of counting. Owing to the high degree of self-absorption of the β-particles in the AgCl crystal, the observed activity arises from the β-particles coming from radioactive silver nuclei present in a layer of the crystal about 0.4 mm thick. The conditions are thus very favourable for detecting a migration of silver ions through distances greater than about 0.4 mm, on the application of a D.C. electric field. Similar irradiations were then carried out with the cooled crystal together with the applied field. The activity of the crystal was then measured as before, arranging that the surface of the crystal nearest the window of the Geiger counter was the surface in contact with the platinum cathode during the time of irradiation. Under these conditions, a migration of the interstitial silver ions in the field over a distance greater than 0.4 mm towards the cathode, would show up in a large increase in the observed activity. If all the radioactive silver ions in the crystal migrated to a region close to the cathode, it was estimated that an initial activity of about 1,400 cts/min would be obtained after irradiation with an applied field*. It was estimated that a migration of the interstitial ions through a distance greater than 0.1 mm would yield a statistically significant increase in the activity.

In fact, no statistically significant increase in the counting rate was observed in a number of trials. A number of different crystals were used and care was taken as far as possible to prevent thermal shocks which might set up strains and dislocations in the crystals. A series of experiments were also carried out with crystals which were slowly annealed, during a period of 24 hours, from 400°C to the operating temperature; but again, no increase of activity was observed after irradiation with an applied field.

DISCUSSION

It is of course not possible to give an unambiguous interpretation to the negative results obtained. It is possible that the crystals used were not sufficiently perfect and that the interstitial radioactive ions were trapped at faults or dislocations in the crystals. However, assuming the postulated mechanism for the production of interstitial ions

* This calculation was carried out taking into account that the concentration of radioactive silver produced by thermal neutron capture near the surfaces of the crystal is somewhat greater than the concentration in the body of the crystal owing to the change in neutron flux throughout the crystal produced by neutron absorption.

by nuclear recoil, and that the conditions (1) and (3) mentioned above are satisfied, it is possible to calculate a lower limit for the mobility of the silver interstitial ions. The negative result then shows that

$$vEt < 0.01 \text{ cm}$$

where v is the mobility of the Ag ions, E is the electrostatic field in the crystal and t is the migration time of the interstitial ions in the applied field. Under the conditions of the experiments, t is effectively the half-life of Ag^{108} (see condition (2) above) — 138 sec, and E is about 1000 volts/cm (taking into account the dielectric constant of AgCl).

One thus obtains that, at −80°C, the mobility v is less than 7×10^{-8} cm^2 sec^{-1} volt^{-1}.

If we write $v = v_0 e^{-U/kT}$ and take $v_0 = 0.12$[4], we obtain a lower limit for U, the activation energy for diffusion: $U > 0.23$ e.v.

This result is consistent with the value of 0.25 obtained by Koch and Wagner. As mentioned in the introduction, however, there were reasons to expect that the value for U might be nearer to the value 0.11 e.v. obtained for AgBr. For U equal to 0.11 e.v. one would expect a large positive effect. It would therefore be interesting to repeat the above experiments with silver bromide crystals. In the case of silver bromide the measurements would be complicated, however, by the bromine activity which would be produced, since bromine has a high capture cross-section for slow neutrons. If strong thermal neutron sources are available, it would be of interest to search for the postulated effect in other ionic crystals.

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THE *M* X-RAY FROM RADIUM *D* AND THE *M* X-RAY FLUORESCENCE YIELD OF BISMUTH

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INTRODUCTION

M x-ray fluorescence yields (i. e. the ratio between the number of x-rays emitted to the number of initial vacancies in the shell) have so far been determined only in the case of uranium, for which the total yield of the *M* shell has been measured by Lay,[1,2] using fluorescent excitation. Lay compared the intensity of the *M* x-rays from a thick specimen of uranium with that of the exciting beam by a photographic method. Similar measurements would be difficult to make for elements of lower atomic number than uranium because of the low intensity of the emitted x-rays relative to that required to excite them. Alternative methods for the investigation of *M* yields are, however, provided in certain cases by the use of radioactive sources. These have already been employed in the determination of *L* yields.[3]

In the experiments to be described it has been found possible to measure the intensity of *M* x-rays from a thin source of radium *D*, by means of a proportional counter. Bismuth x-rays are emitted as a result of internal conversion of γ-rays following the Ra*D* → Ra*E* transition, and the intensity of the *M* x-rays was compared with that of the *L* x-rays from the same source. As will be seen, since the *L* fluorescence yield and the relative amounts of conversion in the shells involved are known, an estimate of the *M* fluorescence yield of bismuth can thus be made.

EXPERIMENTAL METHOD

A source of Ra*D* (strength about 20 microcuries) in equilibrium with its products was used, and in order to obtain good efficiency of detection of the readily absorbed *M* x-rays, it was placed within the glass envelope of the proportional counter (Figure 1). The brass cathode, 3 cm in diameter, has a hole 1 cm in diameter to admit the radiations under examination. A similar hole, opposite a glass bubble window, is available for the purpose of making energy calibrations using external sources. The Ra*D* source is mounted on a polystyrene backing attached to a ground joint, and is 5.0 cm from the counter cathode. Ra*E* β-particles are prevented from entering the sensitive region of the counter by a strong permanent magnet.

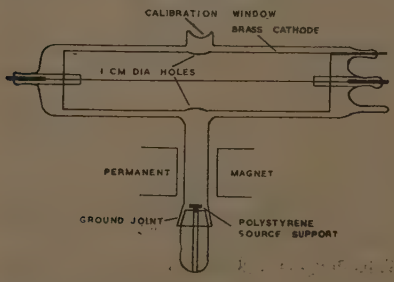

Figure 1

The counter was filled with methane-argon mixtures (~20% methane) and

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although the calculated pressure for optimum efficiency of detection of the *M* x-rays is about 20 cm of mercury, this is not critical and higher pressures were generally employed, leading to better counter stability and reducing wall and "escape" effects. It was, however, necessary to cover the source with a thin layer of polythene (5 mg/cm²) in order to prevent the polonium α-particles from reaching the counter. After suitable amplification, the pulse size distribution was determined by means of a single channel pulse analyser.

RESULTS

The results shown in Figure 2 are from measurements made with the counter filled with 34 cm pressure argon and 8 cm methane. In Figure 2a the pulse size distribution corresponding to energies up to 4 kev (calibrated with 5.9 kev Mn x-rays from an external source of Fe^{55}) is given, and shows the peak attributed to the bismuth *M* x-rays. The maximum of this peak occurs at about 2.5 kev, and since the theoretical semi-half width of a line at this energy is 0.3 kev, the individual *M* x-ray lines are not resolved. In view of the fact that the width of the peak obtained is nearly the same as the theoretical width of a monochromatic line, however, it appears that most of the *M* x-ray quanta emitted have energies within about 0.2 kev of the maximum.

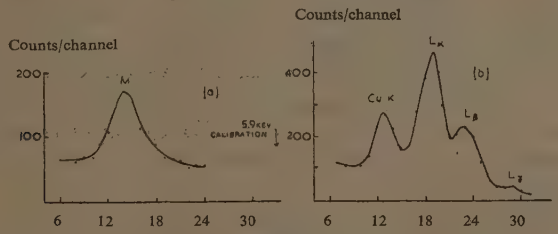

Figure 2

Figure 2b shows the spectrum of *L* x-rays obtained with the same source and counter filling, but at lower amplifications. These *L* x-rays have been previously observed and discussed by several investigators.[5,6,7,8]

In estimating the relative intensities of the radiations, the continuous background (caused by the stopping of β-particles from the source) has been deducted. For the *L* x-rays, corrections have been made to compensate for the "escape peaks" resulting from events in which argon *K* x-rays escape from the counter gas. The magnitude of this correction was determined in a subsidiary experiment, using a source of Fe^{55} with the same counter filling, and was found to be 16%. The estimated average Ra*D* source thickness was ~10 microgm/cm² (0.2 mg of Pb/mc) and the mass absorption coefficient for self-absorption of the *M* x-rays, although not accurately known, must be less than 2000 cm²/g, so that self-absorption should not exceed about 1% and has been neglected.

The observed rates of counting of quanta of the various energies are given in Table I, together with the calculated factors for a) absorption in the polythene filter, b) absorption in the gas between the source and the counter cathode, c) gas efficiency of the counter.

TABLE I

Type of radiation	M x-rays	L_α	L_β
Mean energy (kev)	2.5	10.8	13.0
Fraction of quanta transmitted by polythene	0.532	0.991	0.995
Fraction transmitted by gas between source and counter	0.320	0.814	0.895
Counter gas efficiency	0.498	0.117	0.068
Number of quanta counted per minute	740	2390	1390
Number of quanta emitted within solid angle of counting per minute	8.7×10^3	25.2×10^3	23.0×10^3

CALCULATION OF THE M FLUORESCENCE YIELD

The conversion electron spectrum of RaD has been investigated by Cranberg[9] and by Butt and Brodie[10]; more recently both the emission and conversion spectra have been reexamined by Wu et al[11]. The energy of the excited state of RaE, following β decay of RaD, is 46.7 kev, and internal conversion of resulting γ-rays can occur in the L and subsequent shells only; K conversion being energetically impossible. Conversion lines corresponding to a 46.7 kev γ-ray occur in the majority of disintegrations ($\sim$70%) and the relative amounts of ionisation thus produced in the $L_{\rm I}$, $L_{\rm II}$, $L_{\rm III}$ and M shells is 100:9:2:29 respectively[12]. The total L fluorescence yield for RaE has been found by Kinsey[3] to be 0.47.

The observed M x-rays result from ionisation produced by:

a) Coster-Kronig transitions between the L subshells (in bismuth, $L_{\rm I}\rightarrow L_{\rm III}$ $M_{\rm IV,V}$ are the only transitions of this type involving the M shell which are energetically possible),

b) Auger transitions such as $L\rightarrow MM$, $L\rightarrow MN$ etc.,

c) radiative $L\rightarrow M$ transitions,

d) M conversion of γ-rays.

Assuming initial ionisations of 100, 9, and 2 vacancies in the $L_{\rm I}$, $L_{\rm II}$, $L_{\rm III}$ subshells respectively, corresponding to the emission of 52 L x-rays (total L ionisation $\times$ L fluorescence yield) it is possible to estimate the amount of M ionisation resulting from the above processes.

1) Kinsey[4] has estimated that in bismuth 70% of $L_{\rm I}$ vacancies are filled by Coster-Kronig transitions of the type $L_{\rm I}\rightarrow L_{\rm III}$ M, and experimental confirmation of this figure has been obtained by Salguiero and Valadares[6]. 70 $M_{\rm IV,V}$ vacancies result, therefore, from such transitions and the original L ionisation is redistributed to 30, 9, 72 vacancies in $L_{\rm I}$, $L_{\rm II}$ and $L_{\rm III}$, respectively.

2) After allowing for the emission of $3L_{\gamma_1}$ rays[7], the 52 L quanta resulting from the assumed initial L ionisation may be divided into two main groups, $L_{\alpha_{1,2}}$ and $L_{\beta_{1,2,3,4}}$, numbering about 26 and 23 quanta respectively, in accordance with the relative intensities found (see Table I). The first group derive from $L_{\rm III}\rightarrow M_{\rm IV,V}$ tran-

sitions, and lead to 26 $M_{IV,\,V}$ vacancies. Of the L_β quanta, all derive from transitions to the M shell except L_{β_2} ($L_{III} \rightarrow N_V$). If, however, the ratio of intensities of $\beta_2 : \alpha_1$ is taken as 1:5, as for x-rays excited by electronic bombardment since both rays are from transitions to the same level (L_{III}), a further 18 M ionisations are found to result from the L_β transitions. A total of 44 M vacancies are contributed, therefore, by $L \rightarrow M$ radiative transitions.

3) The 59 (111—52) L ionisations as yet unaccounted for give rise to radiationless Auger transitions. Although no exact theoretical or experimental information is available of the relative rates of the energetically possible L Auger transitions, those involving the M shell predominate. The most probable transition is $L \rightarrow MM$ (giving double M ionisation) and $L \rightarrow MN$ transitions will also be important. It is unlikely, however, that transitions involving only the N or subsequent shells amount to more than a small fraction of the total L Auger rate. If the $L \rightarrow MM$ transition is twice as frequent as $L \rightarrow MN$, 98 M ionisations would result. Adopting this figure with a possible error of ± 20 covers the extreme cases, either that the rates of $L \rightarrow MM$ and $L \rightarrow MN$ transitions are equal, or that of $L \rightarrow MM \gg L \rightarrow MN$, and at the same time up to 9% $L \rightarrow NN$ transitions would be allowed for.

The reorganisation of 111 L vacancies, by the above processes, leads therefore to a total of 212 M ionisations, and to this figure must be added a further 29, the corresponding number of M converted γ-rays, so that in all 241 M vacancies are created for every 52 L x-rays observed. From the results shown in Table I, and allowing for the small contribution of $L\gamma$ as above, 8.9 M x-rays are emitted. The value 0.037 is thus obtained for the total M fluorescence yield of RaE.

CONCLUSION

Although the uncertainty in the number of M ionisations produced by Auger transitions has been assumed to be 20%, only part of the total M ionisation is produced in this way, so that an error of only 8% is thus introduced into the final result. Taking into account this and other possible sources of error, the estimated M yield of RaE may be stated as 0.037 ± 0.007.

When comparing this value for bismuth with the value 0.06 obtained by Lay for uranium, it should be remembered that, in addition to an expected increase of fluorescence yield with atomic number (as with K and L yields), the modes of excitation are different in the two cases, and that in RaE, a large fraction of M x-rays are emitted from atoms which are multiply ionised in the M shell.

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A PROBABILITY APPROACH TO TIES IN RANK CORRELATION

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When rank correlation methods were first proposed by Spearman[1] and Kendall[2], tied ranks were considered merely disturbing, but unavoidable, incidents. In more recent work[3,4], the techniques developed for dealing with tied ranks by Kendall[5] and others were recommended on their own merits for the study of contingency. They have proved to be well suited for measuring the strength of association between two qualitative characteristics of a population, whenever the different values assumed by the characteristics fall into a natural order; if there is no such order among these values, rank correlation methods can obviously not be applied, and if these values are of a quantitative nature, ordinary correlation methods would be preferable.

Various methods of handling tied ranks have been proposed with respect to both Spearman's and Kendall's rank correlation coefficients. With respect to Kendall's coefficient τ, with which *alone* we are concerned in the present study, at least four techniques have been developed; but all of them attribute the same score to a given pair of rankings, and they differ only in the value of a denominator by which the score should be divided in order to be turned into a correlation coefficient. Such a division, though not strictly necessary for the measurement of correlation, yields a coefficient which is more useful than the score itself for comparing the strength of correlation between pairs of rankings with different numbers of members or different types of ties. The relative merits of the four proposed coefficients, with respect to the problem under consideration, will be compared below.

The appearance of ties in a ranking may be due to two different circumstances:

(a) some of the members are identical with respect to the characteristic according to which they should be ranked;

(b) all members could theoretically be ranked according to the characteristic in question, but owing to lack of information, insufficient means of observation or similar sources of error, the rank order of the "tied" members cannot be distinguished.

It might be questioned whether under the former circumstances (a) the use of rank correlation methods is logically justifiable, because according to the very definition of ranking[6] this is the arrangement of ". . . a number of individuals according to a quality which they all possess to a varying degree . . ." and not to the same degree. In the present study, the latter point of view (b) only will be considered, i.e. it will always be assumed that the members of the ranking have theoretical, untied ranks, which shall be called their "true" ranks, and that the ties are due to the failure of the observer to distinguish between some of these true ranks. Thus the appearance of a tie in a ranking is considered as a sort of error. The purpose of the present study is, then, to ascertain

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whether this type of error, namely the random appearance of ties, introduces a bias into the coefficient of rank correlation.

This question can be approached from two different angles:

(a) Assume a pair of untied rankings to be given; what is the effect of tying certain of its members with respect to one or both of the rankings on the rank correlation coefficient? This approach corresponds to the problem of direct probability in sampling theory: what is the distribution of a statistic in different samples drawn from a given population?

(b) Assume a pair of rankings containing ties to be given; which pairs of untied rankings are able to produce the given pair of rankings? This approach corresponds to Bayes problem or inverse, or fiducial, probability in sampling theory: which values of certain parameters are likely to obtain in the population from which a given sample was drawn?

It is interesting to note that only the latter approach has been considered explicitly by Kendall[6]. Kendall showed that the scores which would be obtained from the various untied rankings which may generate a given pair of rankings containing ties are distributed symmetrically with respect to the score computed from the given pair of rankings. On the basis of this fact Kendall concludes that the rank correlation (his coefficient t_a) for the pair of ranks containing ties can be considered ". . . as an average coefficient such as would be obtained if the tied ranks were replaced by integral (untied) ranks in all possible ways, τ (the rank correlation coefficient for untied rankings) be calculated for each and the arithmetic mean taken of the resulting values . . .". This justification for estimating, on the basis of a given pair of rankings containing ties, the rank correlation coefficient which would be observed in the underlying pair of untied rankings, if it was available, introduces tacitly the assumptions that any of the untied rankings which may generate the given rankings containing ties is *a priori* equally probable and that the probability of any pair of untied rankings with score S to generate a pair of rankings containing ties with a score S', is a symmetrical function of the difference $S - S'$. It will appear from the following considerations that the latter assumption may not always be acceptable.

The former problem, the one which corresponds to direct probability approach in sampling, has been very briefly mentioned by Sillitto[7]. But in connection with the purpose of his study he was interested merely in (I) the case where ties occur in one of the two rankings only, a case which is not general enough for the application of rank correlation methods to the study of contingency; (II) ties of order 2 or 3, i.e. pairs (or "duplets") and triplets; (III) the "basic" distribution of scores, i.e. the distribution of scores obtained by assigning the ties in a certain specified way to the rankings.

It therefore appears worth while to follow out the approach to tied ranks which corresponds to the direct probability aspect of sampling, i.e. to study the distribution of scores (or rank correlation coefficients) which would be observed if all possible ties including duplets, triplets, quadruplets, etc., be introduced in all different possible ways into a given pair of untied rankings. This distribution will be called, by analogy with the accepted term of "sampling distribution", the "tying distribution" of the score or the rank correlation coefficient. In the following, results of such a study will be presented with respect to an example only. This method has been chosen because it seems useless to express the tying distribution, in the general case, in symbolic form, since

such an expression becomes most unwieldy, as will be seen from the expression given below for the number of elements. On the other hand, this experimental approach yields satisfactory results for informative purposes.

The number of different ways in which pairs of rankings containing ties can be obtained from a given pair of untied rankings can be expressed as follows. Sillitto[7] showed that the number of ways of attributing a given set of ties, comprising p_2 duplets, p_3 triplets, p_4 quadruplets, . . ., and p_r r-tuplets among n members of a ranking is given by $(p_1+p_2+p_3+p_4+\ldots+p_r)\,!\,/\,p_1\,!\,p_2\,!\,p_3\,!\,p_4\,!\,.\,.\,p_r\,!$ where p_1 is the number of members of the ranking whose ranks are untied, and, evidently, $p_1+2p_2+3p_3+4p_4+\ldots+rp_r=n$. The number of ways of forming ties among the members of one ranking is therefore

$$T_n=\sum_{p_2=0}^{[n/2]}\sum_{p_3=0}^{[n/3]}\ldots\sum_{p_{n-1}=0}^{[n/(n-1)]}(p_1+p_2+p_3+\ldots+p_{n-1})!/p_1!p_2!p_3!\ldots p_{n-1}!$$

all sums extending over those values of p_j only which do not contradict the condition $p_1+2p_2+3p_3+\ldots.+(n-1)\,p^{n-1}=n$, and $[n/j]$ being the integral part of n/j. The highest order of a tie is assumed to be n-1, i.e. the case of all members of a ranking being tied is excluded.

If n is, say, five — and this is the case to be studied in detail—the number of different rankings containing ties is

$$T_5=\sum_{p_2=0}^{2}\sum_{p_3=0}^{1}\sum_{p_4=0}^{1}(p_1+p_2+p_3+p_4)\,!/p_1!p_2!p_3!p_4!$$

$$=(5+0+0+0)\,!\,/5!\,0!\,0!\,0!+$$
$$+(3+1+0+0)\,!\,/3!\,1!\,0!\,0!+$$
$$+(2+0+1+0)\,!\,/2!\,0!\,1!\,0!+$$
$$+(1+2+0+0)\,!\,/1!\,2!\,0!\,0!+$$
$$+(0+1+1+0)\,!\,/0!\,1!\,1!\,0!+$$
$$+(1+0+0+1)\,!\,/1!\,0!\,0!\,1!=15$$

The number of different pairs of rankings containing ties which can thus be generated by a given pair of rankings of n members is obviously T_n^2. This number increases very quickly with n and, for n as small as 8, it approaches already 25,000.

The example to be presented here has been obtained by assuming the $n=5$ members (A, B, C, D, E) of a pair of rankings to be ranked in accordance with their order in one of the rankings (X); this assumption does not impair the generality of the example, because the members can always be arranged in this order. The other rank (Y) has been attributed to the members at random, i.e. by using a table of random numbers. The following pair of rankings has thus been obtained:

	A	B	C	D	E
X	1	2	3	4	5
Y	5	2	3	1	4

The score of this pair of rankings, i. e. the "true" score of our example, is $S=-2$. The number of different pairs of rankings containing ties which this pair can generate is, as shown above, $T_5^2=225$.

The distribution of the scores of these 225 pairs of rankings, which is the tying distribution of this particular case, is shown in Table I. This distribution exhibits the following properties:

TABLE I

The "tying distribution" of scores for an example

Score	Total	−6	−5	−4	−3	−2	−1	0	1	2	3
Frequency	225	2	5	26	35	46	47	35	16	9	4

1) The distribution is asymmetric or skew.

2) The modal score which is, in this case, −1, does not correspond with the true score, i.e. the score of the untied rankings which is −2. In this connection it should be mentioned that the set of ties represented symbolically by $p_1=n$; $p_j=0$ for $j>1$, with respect to both rankings, i.e. the untied ranking, has been included in the tying distribution for reasons of computational convenience. But if it be excluded from the distribution, the frequency of the modal score (−1) exceeds that of the true score (−2) even more. If, on the other hand, scores of pairs in which the ranks of all members are tied with respect to either or both rankings, the number of zero scores is increased by 31 and this score becomes the modal score.

3) The average, or expected, score is −1.59, thus differing also from the true score. This is the most important conclusion to be drawn from this experiment: the score of a pair of rankings containing ties cannot be considered as an unbiased estimate of the score of the pair of true, i. e. untied rankings, if the ties are supposed to have appeared as the consequence of a random process.

4) The highest score obtained is +3. It thus appears that if ties are introduced in a suitable way, fundamentally (i.e. in the absence of ties) negatively correlated characteristics may be made to appear positively correlated. It is obvious that in practical applications, ties are not introduced artificially into a ranked array for the purpose of estimating the correlation; but, on the other hand, they are normally not the result of a random process. The danger of introducing a personal bias (in addition to the bias of estimation mentioned under (3)) must therefore always be reckoned with, when the association of attributes is studied by rank correlation methods.

5) The lowest score obtained is −6. This score differs from the true score of −2 by four units only, while the highest score of +3 differs from the true score by five units. This is only an additional aspect of the skewness of the tying distribution.

6) It appears as if the introduction of ties has the tendency of lowering the absolute value of the score, in addition to spreading the scores over a considerable range. It is probably the combined effect of both these tendencies which reflects itself in the lack of symmetry of the distribution and most of its other characteristics.

As mentioned previously, it is more suitable to measure the correlation by dividing the score by a convenient denominator, than by the score itself. Kendall's rank correlation coefficient for untied rankings (or the true correlation coefficient) is obtained by dividing the score by $n(n-1)/2$. In the present case $n(n-1)/2 = 5(5-1)/2 = 10$, and the true coefficient of correlation is − 0.2. This is the true correlation coefficient with which rank correlation coefficients for rankings containing ties computed by the four different methods mentioned above shall be compared.

a. Kendall's[6] coefficient t_a is the score divided by $(n-1)n/2$. The distribution of this coefficient does not differ from that of the score, and all remarks made with respect to the score apply to this coefficient as well.

b. Kendall's[6] coefficient t_b is the score multiplied by

$$2/\sqrt{[n(n-1)-\sum_r p_r r(r-1)]\ [n(n-1)-\sum_s p_s s(s-1)]}$$

where the sum in r is to be extended over the values of the products $p_r(r-1)r$ with respect to all ties of one of the rankings, and the sum in s over the corresponding products of the other ranking. The distribution of t_b (see Table II) differs considerably from that of the score. It remains skew but to a lesser extent; on the other hand, it is considerably rugged. The arithmetic mean (–0.21) is very near the true value of the correlation coefficient. The mode is not easy to estimate because of the ruggedness of the distribution; but the median (–0.222) is very near the arithmetic mean. The most frequent value of t_b is zero, and none of the coefficients between rankings of which at least one contains a tie assumes exactly the value –0.2. Furthermore, the range of t_b is even larger than that of t_a: it extends from –1 to +0.612.

c. Stuart's[4] coefficient of correlation t_c is obtained from the score by multiplying it by $2m/n(m-1)$ where $m=\sum_{r=1}^{n-1} p_r$ is the number of distinct ranks with respect to that of the two rankings for which this number is smaller. The distribution of t_c (see Table II) resembles that of t_b, but it is even more rugged and appears, in fact, definitely bimodal, with one mode between –0.3 and –0.4 and the other near 0. Thus, there is a hollow just in the region of the true coefficient of correlation, a situation which is not favorable for efficient estimating. But both the arithmetic mean (–.0198) and the median (–0.213) are very near the true coefficient of correlation and the range is narrower than that of t_b.

TABLE II

The tying distribution of t_b *and* t_c *for an example*

Value of the coefficient	Frequency of t_b	Frequency of t_c
Total	225	225
-1.0 — -0.901	1	—
-0.9 — -0.801	1	—
-0.8 — -0.701	8	2
-0.7 — -0.601	15	17
-0.6 — -0.501	11	5
-0.5 — -0.401	20	23
-0.4 — -0.301	24	41
-0.3 — -0.201	35	25
-0.2 — -0.101	46	18
-0.1 — -0.001	—	30
0.0 — 0.099	35	35
0.1 — 0.199	15	16
0.2 — 0.299	3	1
0.3 — 0.399	6	8
0.4 — 0.499	1	4
0.5 — 0.599	3	—
0.6 — 0.699	1	—

d. A rank correlation coefficient proposed by Sillitto[7] is, in its original form, restricted to the case where ties occur only in one of the rankings. But it has been generalized[8] for the general case. This coefficient is obtained by dividing the score by the maximum possible score for the given number of members and the given combination of ties in both rankings considered together. Although the denominator of this coefficient is not easily expressed in symbolic form, it is very simply calculated in practice: the score of a pair of rankings containing ties is, by definition, the difference between the number of members which are ranked, in the two rankings, in the same order and the number of those ranked in inverse order, tied rankings being disregarded; the maximum possible score is therefore simply the sum of these two numbers, and this is the denominator of Sillitto's coefficient as amended by Gabriel, which shall be designated here by t_d.

The distribution of t_d is very irregular (Table III). The most frequent value is -1, a second maximum appears at the value 0, and a third in the region of $-1/3$. The distribution is spread over the whole range of possible values of a correlation coefficient from -1 to $+1$. The arithmetic mean is -0.297 and the median -0.250.

TABLE III

The tying distribution of t_d *for an example*

Value of t_d	*Total*	-1	$-5/7$	$-2/3$	$-3/5$	$-4/7$	$-1/2$	$-3/7$	$-1/3$	$-1/4$
Frequency	225	41	3	2	8	1	15	10	27	12
Value of t_d		$-1/5$	$-1/7$	$-1/9$	0	1/7	1/5	1/3	1/2	1
Frequency		18	20	4	35	4	7	5	6	7

It might be questioned, whether it is possible to draw any general conclusions from the results of the study of one single example; certain results apply obviously to the particular example only; but others may be of general validity. With respect to the present problem, even conclusions of the latter type are of interest: it is, e.g., important to state the fact, that bias of a certain type and size *may* be introduced by ties into a rank correlation, even if the exact circumstances under which this bias will appear or disappear and the factors which influence its size or direction remain unknown. Among the effects which are probably peculiar to the present example, the most important ones are those which result from the smallness of the number of members (5) of the ranking under consideration. Whenever this number is small there is a relatively high number of rankings containing ties of high order (say, r=n-1, n-2) as compared with the number of rankings containing ties of all other orders, and this proportion decreases with increasing n. Pairs of rankings of which at least one contains a tie of high order yield low scores (0, 1, -1, etc.) because the number of members whose ranks can be compared is small. At the same time the rank correlation coefficient t_d, under these circumstances, often assumes the values 1 and -1. Thus, the ruggedness of the distribution of t_d and the relatively high frequency of low, and particularly zero, scores and the corresponding values of the rank correlation coefficients t_b and t_c, are, at least partly, due to the smallness of the number of members of the ranking used as an example, here.

It is, furthermore, possible that the tying distribution of scores, generated by rankings with a sufficiently high number of members shows a lower degree of skewness and perhaps even of bias, i.e. the arithmetic mean of the scores of the pairs of rankings containing ties differs less from the true score, than in the present example.

On the other hand it can already be definitely concluded from the present study that, under certain circumstances, the tying distribution of scores may be considerably skewed and, in estimating the score or the rank correlation coefficient of a pair of untied rankings on the basis of a pair of rankings containing ties, a biased estimate may be obtained.

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SOME CYLINDRICAL GRAVITATIONAL WAVES

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I. FIELD EQUATIONS

The question of gravitational waves in the general theory of relativity is one which is still in need of clarification. Because of the non-linearity of the gravitational field equations, it is difficult in general to obtain solutions corresponding to gravitational waves, and their interpretation is sometimes unclear. Some time ago the general case of cylindrical waves was investigated[1]. The present note deals with some special cases of cylindrical gravitational waves.

It was shown by Weyl[2] and by Levi-Civita[3] that in the case of a static axial-symmetric gravitational field, the solution of the field equations of the general relativity theory for empty space could be reduced to the solution of the Laplace equation for the Newtonian gravitational potential. If we introduce cylindrical polar coordinates ρ, φ, z, and the time t, and take as the expression for the line element

$$ds^2 = e^{2\psi}dt^2 - e^{-2\psi}\rho^2\, d\varphi^2 - e^{2\gamma-2\psi}(d\rho^2 + dz^2), \tag{1}$$

which can be done through a suitable identification of the coordinates, without any further restriction on the geometry of the space, then the field equations reduce to the following:

$$\nabla^2\psi \equiv \psi_{\rho\rho} + (1/\rho)\psi_\rho + \psi_{zz} = 0 \tag{2}$$

$$\gamma_z = 2\rho\,\psi_\rho\,\psi_z, \qquad \gamma_\rho = \rho(\psi_\rho^2 - \psi_z^2). \tag{3}$$

Here a subscript indicates partial differentiation.

If one chooses a suitable solution of Eq. (2) for ψ, one can then solve Eqs. (3) for γ, since these equations are compatible on the basis of (2).

The above procedure can be readily modified so that it can be used for discussing cylindrical waves. What we do is essentially to interchange the roles of z and t. Thus, we take for the line element, in place of (1),

$$ds^2 = e^{2\gamma-2\psi}(dt^2 - d\rho^2) - e^{-2\psi}\rho^2\, d\varphi^2 - e^{2\psi}dz^2. \tag{4}$$

To satisfy the field equations we now take

$$\psi_{\rho\rho} + (1/\rho)\,\psi_\rho - \psi_{tt} = 0 \tag{5}$$

$$\gamma_t = 2\rho\,\psi_\rho\,\psi_t, \qquad \gamma_\rho = \rho(\psi_\rho^2 + \psi_t^2). \tag{6}$$

These equations are essentially the same as those arrived at in reference 1.

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II. PERIODIC WAVES

Let us consider periodic solutions of the wave equation (5) representing monochromatic waves, i.e., having a sinusoidal dependence on t. The solutions of this type are of the form

$$\psi = AJ_0(\omega\rho)\cos(\omega t+\alpha) + BN_0(\omega\rho)\cos(\omega t+\beta), \tag{7}$$

where J_0 and N_0 are Bessel functions of the first and second kind respectively, of order zero[4], and the frequency ω and the other constants A, B, α and β are arbitrary.

As a particular case, let us take a standing wave described by the solution

$$\psi = AJ_0(\omega\rho)\cos\omega t. \tag{8}$$

Substituting into (6), we get (letting a prime denote a derivative with respect to the argument)

$$\left.\begin{aligned}\gamma_t &= -A^2\omega^2\rho J_0(\omega\rho)J'_0(\omega\rho)\sin 2\omega t,\\ \gamma_\rho &= A^2\omega^2\rho\{[J'_0(\omega\rho)]^2\cos^2\omega t + [J_0(\omega\rho)]^2\sin^2\omega t\}.\end{aligned}\right\} \tag{9}$$

Integrating, we get (to within an arbitrary additive constant)

$$\gamma = (1/2)A^2\omega\rho J_0(\omega\rho)J'_0(\omega\rho)\cos 2\omega t + (1/2)A^2\omega^2\rho^2\{[J'_0(\omega\rho)]^2 - J_0(\omega\rho)J''_0(\omega\rho)\} \tag{10}$$

so that both ψ and γ are periodic functions of t.

A solution of this form, free from singularities, would be suitable for describing a situation in which standing waves are set up by reflection at the surface of a large sphere with centre at the origin. If, in (8) and (10), we replaced $J_0(\omega\rho)$ by $N_0(\omega\rho)$, we would obtain a solution with a singularity at the origin. This might be interpreted as describing a standing cylindrical gravitational wave with matter present along the Z-axis.

Since, for large values of ρ, the asymptotic expansions of the Bessel functions are given by

$$J_0(\omega\rho)\simeq(2/\pi\omega\rho)^{1/2}\cos[\omega\rho-(\pi/4)],\quad N_0(\omega\rho)\simeq(2/\pi\omega\rho)^{1/2}\sin[\omega\rho-(\pi/4)], \tag{11}$$

we get an outgoing wave if we take

$$\psi = AJ_0(\omega\rho)\cos\omega t + AN_0(\omega\rho)\sin\omega t, \tag{12}$$

for the asymptotic expansion then has the form

$$\psi\simeq A(2/\pi\omega\rho)^{1/2}\cos[\omega\rho-\omega t-(\pi/4)]. \tag{13}$$

Substituting the expression (12) into (6) and carrying out the integration, one obtains

$$\begin{aligned}\gamma = (1/2)A^2\omega\rho\{&J_0(\omega\rho)J'_0(\omega\rho) + N_0(\omega\rho)N'_0(\omega\rho) + \omega\rho[(J_0(\omega\rho))^2 + (J'_0(\omega\rho))^2 +\\ &+ (N_0(\omega\rho))^2 + (N'_0(\omega\rho))^2] + [J_0(\omega\rho)J'_0(\omega\rho) - N_0(\omega\rho)N'_0(\omega\rho)]\cos 2\omega t +\\ &+ [J_0(\omega\rho)N'_0(\omega\rho) + N_0(\omega\rho)J'_0(\omega\rho)]\sin 2\omega t\} - (2/\pi)A^2\omega t\end{aligned} \tag{14}$$

It will be seen that in the present case the solution for γ contains an aperiodic term in t. That this would happen for a running wave was pointed out in reference 1. The continuous transfer of gravitational energy by such a wave brings about a permanent change in the metric. However, a wave of this kind would have to be excluded on physical

grounds: since the wave carries away energy from the matter on the Z-axis, there must be a change in the motion of the latter in the course of time in consequence of which the solution for ψ cannot remain periodic in t.

III. PULSE WAVES

Now let us consider the case of a cylindrical wave which starts out at the axis as a disturbance of short duration and travels outward from the axis. For this purpose we take for ψ the expression

$$\psi = (1/2\pi)\int_{-\infty}^{t-\rho} f(\beta)\, d\beta/[(t-\beta)^2-\rho^2]^{1/2}, \tag{15}$$

which is a solution of the wave equation[5]. Here $f(t)$ is a function which represents the strength of the wave source on the Z-axis as a function of time and is assumed to vanish for t less than some finite negative value.

Let us take, as the first example, the case in which

$$f(t) = A\delta(t) \tag{16}$$

where A is a constant and $\delta(t)$ is the Dirac delta-function (actually, the limit of a sequence of functions) defined as satisfying the conditions

$$\delta(x) = 0, \quad \text{for } x \neq 0; \qquad \int_{-\infty}^{\infty} \delta(x)dx = 1. \tag{17}$$

In this case one has

$$\psi = 0, \quad t < \rho; \qquad \psi = A/[2\pi(t^2-\rho^2)^{1/2}], \quad t > \rho. \tag{18}$$

It follows readily from (6) that

$$\gamma = 0, \quad t < \rho; \qquad \gamma = A^2\rho^2/[8\pi^2(t^2-\rho^2)^2], \quad t > \rho. \tag{19}$$

The source of the wave is in the form of a sharp pulse. In both ψ and γ there are singularities at the moment when the wave front passes a given point, followed by a "tail" which persists for a long time. A solution of this type would have to be rejected as unphysical because of the singularities.

Instead of the Dirac delta-function, let us now take for the source function

$$\left.\begin{aligned} f(t) &= 0, \quad t<0, \\ &= B, \quad 0<t<T, \\ &= 0, \quad t>T, \end{aligned}\right\} \tag{20}$$

where B is a constant. Carrying out the integration of (15), one finds that

$$\begin{aligned} \psi &= 0, \quad t < \rho, \\ &= \frac{B}{2\pi} \ln \frac{t+(t^2-\rho^2)^{1/2}}{\rho}, \quad \rho < t < \rho+T, \\ &= \frac{B}{2\pi} \ln \frac{t+(t^2-\rho^2)^{1/2}}{t-T+[(t-T)^2-\rho^2]^{1/2}}, \quad t > \rho+T. \end{aligned} \tag{21}$$

Integrating Eq. (6), one obtains

$$\begin{aligned}\gamma &= 0, \quad t < \rho, \\ &= \frac{B^2}{4\pi^2} \ln \frac{\rho}{t^2-\rho^2}, \quad \rho < t < \rho + T, \\ &= \frac{B^2}{2\pi^2} \ln \frac{x^{1/2}+t^2-Tt-\rho^2}{x^{1/2}}, \quad t > \rho + T,\end{aligned} \tag{22}$$

where

$$x = (t^2-\rho^2)\,[(t-T)^2-\rho^2]. \tag{23}$$

In this case it is seen that there are still singularities present, although they are weaker than in the previous example. The singularities arise from the discontinuities in the source function f, which lead to singularities in the derivatives ψ_t and ψ_ρ (although not in ψ itself); it follows that there will be still stronger singularities in the derivatives of γ which bring about a singularity in γ.

These singularities can be removed by taking the source function f to be continuous. If we consider the behaviour in the neighbourhood of $t=0$, we can take, for example,

$$\left.\begin{aligned} f(t) &= 0, \quad t<0, \\ &= Ct, \quad t>0, \end{aligned}\right\} \tag{24}$$

where C is a constant. One finds in this case

$$\begin{aligned}\psi &= 0, \quad t < \rho, \\ &= (C/2\pi)\{t\ln[(t+(t^2-\rho^2)^{1/2})/\rho]-(t^2-\rho^2)^{1/2}\}, \quad t > \rho,\end{aligned} \tag{25}$$

and

$$\begin{aligned}\gamma &= 0, \quad t < \rho, \\ &= \frac{C^2}{4\pi^2}\left[(1/2)(t^2-\rho^2)+(1/2)\rho^2 \ln^2\frac{t+(t^2-\rho^2)^{1/2}}{\rho} - t(t^2-\rho^2)^{1/2} \ln \frac{t+(t^2-\rho^2)^{1/2}}{\rho}\right], \\ & t > \rho.\end{aligned} \tag{26}$$

We see that both ψ and γ are now well behaved at $t = \rho$.

In order to understand the behaviour of the solution for large values of t, we assume that the source function $f(t)$ is such that

$$f(t) = 0, \quad t<0; \qquad f(t) = 0, \quad t>T. \tag{27}$$

If we take $t-\rho \gg T$, then the integral (15) can be approximated by[5]

$$\psi \simeq (1/2\pi\,(t^2-\rho^2)^{1/2}) \int_0^T f(\beta)d\beta, \tag{28}$$

or, if we let

$$A = \int_0^T f(\beta)d\beta, \tag{29}$$

we get for ψ

$$\psi \simeq A/2\,\pi(t^2-\rho^2)^{1/2} \tag{30}$$

as in (18), and therefore for γ,

$$\gamma \simeq A^2\,\rho^2/8\,\pi^2(t^2-\rho^2)^2, \tag{31}$$

to within an additive constant. We see that the "tail" of the wave is free from singularities.

One can conclude from the above that it is possible (by the use of a continuous source function of finite duration) to obtain solutions for ψ and γ which are well behaved, so that they are physically acceptable. However it must be stressed that there still remains open the question of the generation of such waves. What we have referred to as the source function is a mathematical device for enabling us to get solutions of the field equation with a particular kind of behaviour near the Z-axis. From the physical point of view one would have to replace the Z-axis by a tube of small but finite cross-section in which there is a given distribution of matter described by the components of the stress-energy-momentum density tensor. The source function would be expressed in terms of integrals involving these components. However, these components are subject to certain conditions which represent the equations of motion of the material medium, and the waves emitted would have to be of a form compatible with these conditions. This question requires further investigation.

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ON THE SEMI-CONTINUITY OF THE TRANSFINITE DIAMETER

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The transfinite diameter $\tau(s)$ of an infinite compact set S of points in the complex z-plane is a set-function whose properties have been much investigated since its first introduction[3], some three decades ago, by the author. The present note is a new contribution to the study of variation of $\tau(s)$ when S is subject to a continous variation.

The fact that the transfinite diameter of the ρ-neighbourhood $S(\rho)$ of an arbitrary set S tends to $\tau(s)$ as $\rho \downarrow 0$ is a simple consequence of the original definitions[4]. The counterpart of this semi-continuity (from above), a phenomenon which is more difficult to recognize, is the objective we now pursue. Some special categories of compact sets S, described in detail in the sequel (cf. Theorem I and II), will be considered with the aim of showing the possibility of their approximation (from within) by lemniscates L such that the ratio $\tau(s)/\tau(L)$ is as near to 1 as we like.

1. LEMMA. *Let* D *be a bounded simply connected domain in the* $z = x + iy$*-plane and* C *its boundary. Let* $\rho > 0$ *be given arbitrarily. Then there is a lemniscate*

$$L : |p(z)| = c \qquad (1)$$

$$p(z) = z^m + p_1 z^{m-1} + \ldots + p_m ; \quad c = r^m ; \quad r > 0$$

in D *of radius* r *which is an analytic Jordan-contour and such that the* ρ*-neighbourhood* $L(\rho)$ *of* L *wholly contains* C.

In other words: The boundary C of a simply connected bounded domain D can be approximated, as closely as we please, by lemniscates L of simple connectivity, contained in D.

2. *Proof.* Denote by D^ε the aggregate of all those points of D whose distance from C is $> \varepsilon > 0$. D^ε is *not empty* provided that $0 < \varepsilon < \varepsilon^* = \varepsilon^*(D)$, the upper bound of the radii of all circumferences whose interior entirely belongs to D and it yields an open pointset whose every boundary point lies at a distance ε from C.

Denote by B^ε the boundary of D^ε. I assert: $B^\varepsilon(\rho)$ contains C provided $0 < \varepsilon < \varepsilon_0(\rho) \leqslant \leqslant \varepsilon^*$. In order to verify this assertion, take into consideration the following facts:

1°. By the Heine-Borel theorem, there is a finite set, say $\{P_1, \ldots, P_n\}$ of (distinct) points of C such that $P_1(\rho/3) + \ldots + P_n(\rho/3)$ covers C;

2°. $P_\nu(\rho/3)$ contains, for $1 \leqslant \nu \leqslant n$, a point Q_ν of D;

3°. Whenever $1 \leqslant \nu \leqslant n$, there is a point U_ν on C whose distance from Q_ν equals the distance δ_ν of Q_ν from C;

4°. The segment $Q_\nu U_\nu$ contains a point V_ν at a distance $\varepsilon_0 = \varepsilon_0(\rho) = {}^1/_2 \operatorname{Min}_{1 \leqslant \nu \leqslant n} \delta_\nu$ both from U_ν and C;

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5°. V_ν lies on the boundary B^{ε_0} of D^{ε_0} (indeed, V_ν is a limit point of points in D at a distance $>\varepsilon_0$ from C);

6°. $0<\delta_\nu\leqslant \rho/3$ for $1\leqslant\nu\leqslant n$;

7°. Each point Π_ν of C, contained in $P_\nu(\rho/3)$, lies at a distance $\leqslant \rho$ from V_ν; (in fact, $\Pi_\nu V_\nu\leqslant \Pi_\nu Q_\nu + Q_\nu V_\nu \leqslant 2/3\ \rho + \delta_\nu\leqslant \rho$.

Thus C is contained in $B^{\varepsilon_0}(\rho)$.

3. Now we notice that D^ε necessarily contains B^{ε_0} whenever $0<\varepsilon<\varepsilon_0$; hence C is contained in $B^\varepsilon(\rho)$ as well, provided $0<\varepsilon<\varepsilon_0$.

In fact, if $P\in C$ and $P^*\in B^{\varepsilon_0}$ is a nearest point on B^{ε_0} to P then $PB^\varepsilon\leqslant \rho$; but the rectilinear segment PP^* contains a point $\in B^\varepsilon$ since P and P^* lie respectively at a distance 0 and $\varepsilon_0>\varepsilon$ from C. Hence $P\in B^\varepsilon(\rho)$ as asserted.

4. If B^ε is reduced to a *single* continuum then the existence of a lemniscate (1) with the properties required in the Lemma can be established as follows: We apply the Hilbert[1]-Fekete[2] theorem about the possibility to approximate to B^ε by a (simply connected) lemniscate L with all its foci on B^ε so that it is contained in D and contains B^ε in its interior; thus it separates B^ε from C. Hence C is contained in $L(\rho)$. Indeed, if Π is an arbitrarily given point of C, there is a point Π^ε on B^ε at a distance $\leqslant \rho$ from Π; as the point of intersection of the segment $\Pi\,\Pi^\varepsilon$ with L lies at a distance $\leqslant \rho$ from Π, $\Pi\in L(\rho)$.

5. If B^ε is *not* reduced to a *single* continuum then it can be wholly enclosed (again by the Hilbert-Fekete proposition) by a lemniscate $L^*\in D$ (consisiting of a *finite* number of analytical Jordan contours mutually exterior to each other) separating the closed set $B^\varepsilon+D^\varepsilon$ from C. The branches of L^* containing $B^\varepsilon+D^\varepsilon$ in their interiors can be connected pairwise, outside D^ε and inside D, by Jordan arcs so that the continuum K composed from these branches of L^* and the Jordan arcs just mentioned lie entirely in D and outside D^ε. Now the Hilbert-Fekete theorem ensures the possibility to approximate to K by a (simply connected) lemniscate L with all its foci on K so that it is contained in D and contains K in its interior thus separating B^ε from C. This L possesses all the required properties. Thus the proof of our Lemma is completed.

6. In possession of this result we proceed to establish the fact that the transfinite diameters $\tau(L)$ of the lemniscates (1) approximate that of the boundary C of the domain D of the Lemma as closely as we please. More precisely we want to verify the following

THEOREM I: *Let* D *be a bounded simply connected domain in the* z*-plane and* C *its boundary. Let* $\eta>0$ *be given arbitrarily. Then there is a* (*simply connected*) *lemniscate* (1) *interior to* D *such that*

$$1\leqslant \tau(C)/\tau(L)\leqslant 1+\eta. \tag{2}$$

7. *Proof.* Let us consider, together with a lemniscate (1) satisfying the requirements detailed in our Lemma, also the *smallest* confocal lemniscate

$$\Lambda: |p(z)| = \gamma;\quad \gamma=\sigma^m,\ \sigma>r \tag{3}$$

containing C thus passing through a point $M\in C$. Then

$$z^* = \sqrt[m]{p(z)}, \tag{4}$$

(the m-th root being *suitably* determined) is, on and outside L, a regular single valued and univalent function of z, mapping (conformally) the exterior of L into $|z^*|>r$ on

the z^*-plane. Moreover $|z^*| = \sigma$, the image of (3) on that plane under the same transformation, encloses the image C^* of C by (4) and passes through a point M^* which corresponds to M by (4) on the z^*-plane. By our supposition concerning L, M lies at a distance $\delta \leqslant \rho$ from L, thus there is a segment MN of length δ joining M with a point $N \in L$ *outside* L. To MN corresponds, by (4), a Jordan arc A in the z*-plane joining M^* with a point N^* of $|z^*| = r$, the image of L on the z^*-plane by (4).

8. This arc may have points of intersection with $|z^*| = \sigma$ different from M^*. We call P^* the first point of A at which A meets $|z^*| = \sigma$ when it starts at N^*; thus the subarc A^* of A joining N^* with P^* lies entirely in the ring $r \leqslant |z^*| \leqslant \sigma$. Now since $z^* = \sqrt[m]{p(z)}$ maps the domain in the z-plane whose boundary consists of L and $\overline{NM}$ onto a domain in the z^*-plane bounded by $|z^*| = r$ and A, we have, by a Theorem of Fekete[3],

$$\tau(L + \overline{MN}) = \tau(|z^*| = r + A)$$

whence, in view of $A^* \subseteq A$ and by the *monotonicity law*[4] of the transfinite diameter,

$$\tau(L + \overline{MN}) \geqq \tau(|z^*| = r + A^*).$$

9. Combine this with the known fact that of all continua interior to $|z^*| = \sigma$ and containing $|z^*| \leqslant r$ and, at least, one point of $|z^*| = \sigma$, the continuum consisting of $|z^*| = r$ and of the segment joining $z^* = r$ with $z^* = \sigma$ has the smallest transfinite diameter which equals

$$(\sigma + r)^2/4\sigma.$$

We obtain

$$\tau(|z^*| = r + A^*) \geqq (\sigma + r)^2/4\sigma;$$

thus also

$$\tau(L + \overline{MN}) \geqq (\sigma + r)^2/4\sigma.$$

On the other hand, by using the *law of subadditivity* of the transfinite diameter, leading in the case under consideration to:

$$\frac{1}{\log[d_2(L+\overline{MN})/\tau(L+\overline{MN})]} \leqslant \frac{1}{\log[d_2(L+\overline{MN})/\tau(L)]} + \frac{1}{\log[d_2(L+\overline{MN})/\tau(\overline{MN})]}$$

($d_2(x)$ means the span of x)

we get, in view of $d_2(L) \leqslant 4\tau(L)$, $d_2(L + \overline{MN}) \leqslant 4\tau(L + \overline{MN})$,

$$\tau(L) = r, \quad \tau(\overline{MN}) = {}^1/_4\overline{MN} = \delta/4 \leqslant \rho/4,$$

the inequality

$$\tau(L + \overline{MN}) : \tau(L) \leqslant e^{12/\log(1/\delta)} = 1 + o(\delta).$$

Hence we have

$$(\sigma/r) \leqslant 1 + 2\,o(\delta) + 2\sqrt{o(\delta) + o(\delta)^2}.$$

Thus

$$\tau(C)/\tau(L) \leqslant \tau(A)/\tau(L) \leqslant 1 + 2\,o(\delta) + 2\sqrt{o(\delta) + o(\delta)^2}$$

as we asserted.

10. Theorem I is a special case of the following general proposition:

THEOREM II. *Let* $\Delta \ni \infty$ *a multiply connected domain in the* z*-plane and* Γ *its boundary. Suppose that the complement of* $\Delta + \Gamma$ *consists of a finite set of disjoint simply connected domains, say* D_K, $1 \leqslant \kappa \leqslant k$ *with respective boundaries* C_K. *Then given* $\varepsilon > 0$ *arbitrarily, there is a lemniscate* L *decomposed into* k *branches, say* $L^K, 1 \leqslant \kappa \leqslant k$, L^K *lying interior to* D_K *and such that*

$$1 \leqslant \tau(\Gamma)/\tau(L) \leqslant 1 + \varepsilon.$$

11. The proof is based on the possibility (by Theorem I) of finding k (simply connected) lemniscates, say L_K, $1 \leqslant \kappa \leqslant k$, interior to D_K, such that

$$1 \leqslant \tau(C_K)/\tau(L_K) \leqslant 1 + \varepsilon_K \leqslant 1 + \varepsilon^2/4(1+\varepsilon),\ 1 \leqslant \kappa \leqslant k$$

whence the statement readily follows by an application of the Hilbert-Fekete theorem and some elementary facts on conformal mapping quoted and used in § 9. Indeed, let $p_K(z) = z^{m_K} + \ldots$ be the polynomial which occurs in the equation $|p_K(z)| = r_K^{m_K}$ of the lemniscate L_K, $1 \leqslant \kappa \leqslant k$, just mentioned. Let, further, $p(z) = z^m + \ldots$ be a polynomial such that the lemniscate L, defined by $|p(z)| = r^m$ decomposes into k branches L_K, $1 \leqslant \kappa \leqslant k$, lying interior to D_K and surrounding L_K. (The existence of such $p(z)$ and r is ensured by the Hilbert-Fekete proposition). Then, by the principle of maximum,

$$\operatorname*{Max}_{z \in C_K} |p(z)|^{1/m} \leqslant \operatorname*{Max}_{z \in L_K} |p(z)|^{1/m} \cdot \operatorname*{Max}_{z \in C_K} |p_K(z)|^{1/m_K} \,|\, \operatorname*{Max}_{z \in L_K} |p_K(z)|^{1/m_K} \leqslant r\, \rho_K \,|\, r_K .$$

Here $\rho_K = \operatorname*{Max}_{z \in C_K} |p_K(z)|^{1/m_K}$ is the radius (=transfinite diameter) of the smallest lemniscate Λ_K, confocal with L_K and surrounding C_K; hence $\tau(C_K) \geqq (\rho_K + r)^2/4\rho_K$. The last inequality, combined with $\tau(C_K) \leqslant (1+\varepsilon_K)\tau(L_K) = (1+\varepsilon_K)\, r_K$ yields, in view of $\varepsilon_K \leqslant \varepsilon^2/4(1+\varepsilon)$, $\rho_K : r_K \leqslant 1 + 2\varepsilon_K + 2\sqrt{\varepsilon_K + \varepsilon_K^2} \leqslant 1 + \varepsilon$. Thus $\tau(\Gamma) \leqslant \leqslant \operatorname*{Max}_{z \in \Gamma} |p(z)|^{1/m} \leqslant r(1+\varepsilon) = \tau(L)(1+\varepsilon)$, as required.

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SOME IMPLICATIONS OF A SMALLER NUCLEAR RADIUS

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INTRODUCTION

The shell model of the nucleus[1,2,3] has been successful in interpreting the observed regularities in nuclear moments and the discontinuities in binding energies. It has not been so successful in quantitative calculations of quadrupole moments and in the interpretation of the fission process. Townes, Foley and Low[4] have pointed out that observed quadrupole moments are much larger than can be accounted for by the single particle model of the nucleus. These large quadrupole moments seem to indicate a collective deformation of the nucleus.

A combination of the single particle model and the liquid drop model of the nucleus[5,6] has been explored by Bohr and Mottelson[7,8,9,10] and Hill and Wheeler[11]. Such a "collective model", a shell structure capable of collective motion, seems to incorporate both single particle features and collective behaviour. It can account for regularities of nuclear moments, first excited states of even-even nuclei[12], gamma ray transition probabilities,[9] and isotope shift anomalies[13].

Quantitative calculations of quadrupole moments, however, of even so simple a system as one nucleon outside a closed shell, yield values much larger than those observed. Similar quantitative discrepancies are found in gamma transition probabilities, distortion factor of the nucleus, energies of the first excited states of even-even nuclei, and isotope shift anomalies.

In all these calculations the nuclear radius enters as a parameter. In some cases these calculations are very sensitive to small changes in the value of the nuclear radius. It will be shown that if a reduced radius of $R=1.2\times10^{-13}A^{1/3}$ cm is assumed, instead of the customary value of $R=1.5\times10^{-13}A^{1/3}$ cm, the agreement between theory and experiment is considerably improved.

So small a radius has been found recently by Fitch and Rainwater[14,15] in a study of X-rays from mesonic atoms. High energy electron scattering experiments by Hofstadter, Fechter, and McIntyre[16] and the interpretation of the results by Schiff[17] seem to confirm a considerably smaller radius than 1.5×10^{-13} cm. They find, in addition, that the charge distribution is not uniform and that the best interpretation of their results indicates a peaked charge distribution in the centre, which slowly tapers off towards the edge of the nucleus.

In section I it will be shown that such a reduced radius gives improved values of quadrupole moments, M-4 transition probabilities, ratio of distortion factor as deduced from quadrupole moments to that deduced from first excited states of even-even nuclei, and isotope shift anomalies. In section II the implications of a non-uniform charge distribution are explored. For a detailed account of the collective model the reader is

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referred to the papers by Bohr and Mottelson[10] and Hill and Wheeler[11], on which these calculations are based.

I. CALCULATIONS ON THE BASIS OF A RADIUS OF 1.2×10^{-13} $A^{1/3}$ CM

A. Quadrupole moments

In the collective model the assumption of a uniform charge distribution and a sharp nuclear boundary are retained. According to this model the nucleus is able to perform oscillations and rotations in a manner similar to molecules. The excitation of the lowest vibrational energy E of such a spherical drop has been calculated classically by Rayleigh[18] and is given in the quantum analogue by

$$E = hw/2\pi = (C/B)^{1/2} \quad (1)$$

where $C = C_{surface} - C_{coulomb}$

$$= 4R^2S - 3/10\,\pi(Z^2e^2/R) \quad (2)$$

Bohr[10] assumes that the dynamic surface tension S is equal to that in the static semi-empirical mass formula of Bohr and Wheeler[4] and is given by $4R^2S \sim 15A^{2/3}$ Mev. B is a measure of the mass taking part in the vibration and is given for a uniform sphere by $B=3/8\,\pi\,(AMR^2)$ (3)

where M is the nucleon mass and A the atomic number.

The nucleons outside the shell are coupled to the surface, and this particle-surface interaction is a cause of the deformation of the surface. Following Bohr we use the dimensionless parameter

$$x = (5/16)^{1/2}(k/j)(hwC/2\pi)^{-1/2} \quad (4)$$

as a measure of the strength of the coupling. The constant k is proportional to the interaction energy. A more detailed discussion of k is given in the appendix. Implicit in this formulation is that j remains a good quantum number.

In the collective model the quadrupole moment consists of two contributions, that of the particles outside the shell Q_{sp} and that due to the deformation of the surface Q_s

$$Q_L = Q_{sp} + Q_s \quad (5)$$

Q_s is a function of the coupling parameter x. We shall limit the discussion to one single particle outside a closed shell or a single hole in a shell. The QM (quadrupole moment) then is given by

$$Q_s = Q_0 P_Q(x) \quad (6)$$

where Q_0 is the intrinsic quadrupole moment and is given by

$$Q_0 = 3/5\,\pi\,(ZR^2) < \beta\cos\gamma > \quad (7)$$

$$= -3/4\,\pi\,[(2I-1)/2(I+1)](k/C)ZR^2 \quad (8)$$

β and γ are distortion coordinates of the surface relative to the nuclear axes. $P_Q(x)$ is a projection factor and a function of the coupling.

As seen from equation (8) the intrinsic quadrupole moment is proportional to the square of the nuclear radius. A smaller radius will therefore give an appreciably smaller

TABLE I

Comparison of experimental quadrupole moments with collective and single particle esttimates

1	2	3	4	5	6	7	8	9	10	11	12	13	14
Nucleus	I	Q_o	x	Q_{sp}	Q_L	Q_o^c	x'	x''	Q_{sp}^c	Q'_L	Q''_L	$Q_L^{(st)}$	Q_{obs}
		$R=1.44\cdot 10^{-13}A^{1/3}$cm				$R=1.2\cdot 10^{-13}A^{1/3}$cm							
$^{11}_{5}B$	3/2	+0.07	0.71	+0.023	+0.06	+0.05	0.71	0.98	+0.016	+0.05	+0.04	+0.03	+0.06
$^{17}_{8}O$	5/2	−0.20	0.56	−0.0013	−0.16	−0.14	0.56	0.73	−.009	−0.11	−0.08	−0.05	−0.005
$^{27}_{13}Al$	5/2	+0.32	0.56	+0.007	+0.30	+0.22	0.56	0.73	+0.045	+0.22	+0.17	+0.12	+0.16
$^{33}_{16}S$	3/2	−0.31	0.73	0	−0.22	−0.21	0.74	1.02	0	−0.14	−0.12	−0.04	−0.08
$^{35}_{16}S$	3/2	+0.31	0.73	0	+0.22	+0.21	0.74	1.02	0	+0.14	+0.12	+0.04	+0.06
$^{35}_{17}Cl$	3/2	−0.32	0.73	−0.055	−0.26	−0.22	0.74	1.03	−0.038	−0.18	−0.16	−0.08	−0.084
$^{37}_{17}Cl$	3/2	−0.32	0.73	−0.055	−0.26	−0.22	0.74	1.03	−0.038	−0.176	−0.156	−0.08	−0.066
$^{63}_{29}Cu$	3/2	−0.61	0.76	−0.08	−0.48	−0.46	0.80	1.10	−0.06	−0.36	−0.32	−0.155	−0.13
$^{65}_{29}Cu$	3/2	−0.61	0.76	−0.08	−0.48	−0.46	0.80	1.10	−0.06	−0.36	−0.32	−0.15	−0.12
$^{69}_{31}Ga$	3/2	+0.67	0.77	+0.08	+0.53	+0.50	0.80	1.10	+0.06	+0.38	+0.34	+0.18	+0.24
$^{71}_{31}Ga$	3/2	+0.67	0.77	+0.08	+0.53	+0.50	0.80	1.10	+0.06	+0.38	+0.34	+0.18	+0.15
$^{73}_{32}Ge$	9/2	−1.3	0.45	0	−1.2	−0.98	0.47	0.63	0	−0.84	−0.75	−0.53	−0.20
$^{113}_{49}In$	9/2	+2.4	0.51	+0.21	+2.4	+1.90	0.55	0.76	+0.15	+1.71	+1.45	+1.15	+1.18
$^{119}_{49}In$	9/2	+2.4	0.51	+0.21	+2.4	+1.89	0.55	0.76	+0.15	+1.66	+1.43	+1.12	+1.20
$^{121}_{51}Sb$	5/2	−2.1	0.68	−0.17	−1.5	−1.70	0.73	1.01	−0.12	−1.15	−0.95	−0.73	−1.00
$^{123}_{51}Sb$	7/2	−2.4	0.58	−0.20	−2.1	−1.90	0.63	0.86	−0.14	−1.47	−1.29	−1.04	−1.20

The table compares measured quadrupole moments, Q_{obs}, of nuclei with one nucleon outside, or a hole inside, a closed shell. Column 2 lists spin of the nuclei. Columns 3, 4, 5, and 6 refer to intrinsic quad. moment Q_o, coupling factor x, quad. moment calculated in single particle model Q_{sp} and the collective model quadrupole moment Q_L respectively. The values are from Bohr & Mottelson[10], using a radius $R=1.44\times10^{-13}$ cm $A^{1/3}$. Columns 7 to 13 refer to calculations with radius $R=1.2\times10^{-13}$. Columns 7 and 10 are the respective intrinsic quad. moment Q_o^c and single particle quad. moment Q_{sp}^c. Columns 8 and 9 show coupling factors x' and x'' for which the collective model quad. moments Q'_L, Q''_L have been calculated (columns 11, 12) Column 13 gives the quad. moment for strong coupling and column 14 the observed quad moments.

quadrupole moment. Table I shows the results of the calculations based on the equations given in the appendix (B).

Columns 3, 4, 5, and 6 refer respectively to Q_0 as given by (8), x the coupling parameter based on $k=40$ Mev (see appendix), Q_{sp} the single particle contribution to QM, and Q_L the total $Q.M.$, all calculated on the basis of a radius of $1.45\times10^{-13}A^{1/3}$ cm. Columns 11, 12, 13, 14 refer respectively to Q'_L using coupling x', Q''_L using $x''=1.3x'$ (see appendix), Q_L (st) based on the strong coupling approximation, all calculated on the basis of a radius of 1.2×10^{-13} cm and Q_{obs} giving the observed quadrupole moments. The ratio of quadrupole moments of two isotopes is a much more reliable experimental result than the individual quadrupole moments.

The general conclusions which can be drawn are as follows: a) The single particle estimates Q_{sp} and Q'^{c}_{sp} are too small, even for a single particle outside a closed shell. b) In general, the collective model estimate gives values which are too big. A smaller radius gives very much improved values. This is due in part to the reduced intrinsic moment $Q_0(1.2/1.45)^2$ and in part because of the increase in coupling. c) An intermediate to strong coupling seems to give the best overall results. Still better agreement would be obtained with a radius of 1.0 to $1.1\times10^{-13}A^{1/3}$ cm.

B. Gamma transitions

The lifetime of nuclear isomers is a function of the multipolarity of the transition. The transition probabilities τ_E, τ_M, the inverse of the lifetime, are a very sensitive function of the nuclear radius[19], since

$$\tau_E \sim (R)^{2L}$$
$$\tau_M \sim (R)^{2L-2}$$

where L is the multipolarity in the electric or magnetic transition. In transitions which involve a high spin change, a small change in R will greatly influence the calculated lifetime. A large number of long-lifed isomers are of the type $M4$ (magnetic transitions with spin change of 4 and change of parity). The experimental data have been summarized by Goldhaber and Sunyar[20]. Table II shows the ratio F of the observed transition probability to that calculated on the basis of the single particle model using the larger radius (F) and the smaller radius (F') respectively.

It is seen that while F is of the order of 0.07, a smaller radius improves the value considerably. The average F'_{aver} and the mean deviation from the average are given in Table III. The very close agreement between theory and experiment in these transitions indicates the striking success of the role of the single particle model. According to the collective model these features are retained. The somewhat low F' factors can be explained by a weak coupling of particle to surface.

A similar improvement, by a factor of 4, with the smaller radius is also found in $E3$ transitions. The experimental results, however, show a much larger scatter but are all lower than the theoretical transition probability estimates.

C. Excited states of even-even nuclei

Ford[12] has shown that the level order of excited states of even-even nuclei can be explained by the collective model. The physical reasoning underlying these calculations is that the rotational energy ($E_{rot}=[h/2\pi\,\mathrm{J}]\,[I(I+1)]$ where J is $3\beta_E^2\,B$) depends pri-

marily on the nuclear distortion parameter β_E. This distortion is, in turn, influenced by the orbits of the single particle outside the shell.

Quantitatively Ford has shown that, in the strong coupling approximation, the computed nuclear distortions β_E are too large and the level spacing too small. The calculations depend rather strongly on the moment of inertia J of such a distorted nucleus.

TABLE II

M-4 transitions in odd—A nuclei

Nucleus	F	F'	Nucleus	F	F'	Nucleus	F	F'
	$g_{g/2}$ — $p_{1/2}$			$h_{11/2}$ — $d_{3/2}$			$i_{13/2}$ — $f_{5/2}$	
$^{69}_{30}$Zn	.060	.23	$^{117}_{50}$Sn	.120	.45	$^{195}_{78}$Pt	.049	.19
$^{85}_{36}$Kr	.101	.38	$^{119}_{50}$Sn	.234	.90	$^{197}_{78}$Pt	.053	.20
$^{87}_{38}$Sr	.089	.34	$^{121}_{52}$Te	.082	.31	$^{197}_{80}$Hg	.048	.18
$^{87}_{39}$Y	.055	.22	$^{123}_{52}$Te	.086	.33	$^{199}_{80}$Hg	.054	.20
$^{89}_{39}$Y	.098	.37	$^{125}_{52}$Te	.094	.36	$^{207}_{80}$Hg	.084	.32
$^{91}_{39}$Y	.037	.14	$^{127}_{52}$Te	.080	.31			
$^{89}_{40}$Zr	.092	.35	$^{129}_{52}$Te	.156	.59			
$^{91}_{41}$Nb	.73	2.8	$^{131}_{52}$Te	.236	.90			
$^{95}_{41}$Nb	.061	.23	$^{129}_{54}$Xe	.056	.21			
$^{97}_{41}$Nb	.024	.09	$^{131}_{54}$Xe	.086	.33			
$^{95}_{43}$Tc	.046	.17	$^{133}_{54}$Xe	.105	.39			
$^{97}_{43}$Tc	.046	.17	$^{135}_{54}$Xe	.116	.44			
$^{99}_{43}$Tc	.057	.22	$^{133}_{56}$Ba	.054	.21			
$^{113}_{49}$In	.038	.144	$^{135}_{56}$Ba	.121	.46			
$^{115}_{49}$In	.044	.165	$^{137}_{56}$Ba	.085	.32			

The factor F gives the ratio of observed transition probability to that calculated on the single particle model using $R = 1.45 \times 10^{-13}$ cm. F' is this ratio using a value $R = 1.2 \times 10^{-13}$ cm. The experimental values are taken from Goldhaber and Sunyar[20] (also Bohr and Mottelson[10]).

TABLE III

	F'_{aver}	$\Delta F'_{aver}$
$g_{9/2} - p_{1/2}$	.23	0.07
$h_{11/2} - d_{3/2}$	.41	0.15
$i_{13/2} - f_{5/2}$	.21	0.04

A more general correlation has been found by Ford between the energies of the first excited states of even-even nuclei, interpreted as rotational states, and the neighbouring quadrupole moments of odd A nuclei. A quantitative comparison of β_E (the distortion as calculated from the excited energy levels) and β_Q (calculated from observed quadrupole moments $Q_{obs} = 3/(5\pi)^{1/2} ZR^2[I(2I-1)/(I+1)(2I+3)]\beta_Q$ differs by a factor of about 1.7 for large distortions and by a factor of about 3 for small distortions. A radius of 1.2×10^{-13}, however, would reduce this discrepancy, since $\beta_Q \sim 1/R^2$ and would be increased by a factor of 1.4. A similar decrease would be obtained in β_E.

D. Isotope shift anomalies

There are two main effects which give rise to atomic isotope shifts, namely the finite mass and the extended charge distribution of the nucleus. For heavy elements the nuclear mass shifts are negligible. Racah[21] and Rosenthal and Breit[22] have investigated the effect of an extended nucleus on the isotope shift. For a uniform charge distribution they find that the perturbed energy ΔE is

$$\Delta E = 3\ DR^{2\rho}/[2\ \rho(2\rho+1)(2\rho+3)] \qquad (9)$$

$$\rho = (1-Z^2\alpha^2)^{1/2}$$

$$D = 8\ \pi Ze^2(2Z/a_H)^{2\rho-2}\ (\rho+1)/[\Gamma(2\rho+1)]^2[\Psi^2(0)]$$

where a_H is the Bohr radius, α the fine structure constant and $\Psi(0)$ the non-relativistic wavefunction at $R=0$. The isotope shift, i.e. the difference in ΔE ($\delta\Delta E$) between two isotopes of the same element, is given by

$$\delta\Delta E = [3DR^{2\rho}/(2\rho+1)(2\rho+3)](\delta R/R) \qquad (10)$$

It is this difference which is measured experimentally. Crawford and Schawlow[23] have analyzed the experimental data on the basis of a radius of 1.5×10^{-13} cm. They find that a) the observed shifts are, on the average, smaller by a factor of 0.5 than the shifts predicted on the basis of the volume effect (Eq. 10), b) the shifts seem to show shell structure regularities. Wilets, Hill and Ford[13] have shown that these regularities can be explained as due to nuclear distortions. The effect of nuclear distortion is small on the perturbed energy but appreciable on the differential shift $\delta\Delta E$. Assuming deformations calculated from observed quadrupole moments or from excited states of even-even nuclei, these workers fit the calculated variation of $\delta\Delta E$ with N to the experimental curve. But even if the nuclear distortion effect is taken into account, it is still found that the predicted isotope shift is too large by a factor of about two. Wilets et al. explain this as due to the compressibility and polarizability of the surface of the nucleus, assuming a proton density larger at the surface than in the interior.

Quite obviously a change in the nuclear radius of about 20% would decrease the calculated value obtained by Eq. (10) by a factor of 1.5 and would bring the theoretical estimates and experimental results in close agreement*. The remaining 10—15% difference is probably due to the polarization of the nucleus. Breit[24] has estimated this effect to be of the order of 10—20%.

* Bitter and Feshbach in a recent note[27] come to a similar conclusion. They do not, however, take into account deformation effects, or polarization of the nucleus.

E. Semi-empirical mass formula

Nuclear masses can be approximately calculated by the semi-empirical mass formula developed by Weizsacker[24] and Bohr and Wheeler[6]. In its most recent version[25] it is given by

$$M(A,Z) = 1.00898\,A - 0.00085\,Z - 0.01507\,A + 0.014\,A^{2/3} + 0.083\,(A/2-Z)^2/A + \\ + 0.000627\,Z^2/A^{1/3} + \delta\,(A,Z) \qquad (11)$$

where $\delta(AZ) = \mp 0.036\,A^{-3/4}$ (– for A even, Z even; + for A odd, Z either), the 4th term refers to the surface tension effect and the 6th term to the coulomb effect using the classical energy of a uniformly charged sphere of radius R, i.e.

$$U = 3Z^2e^2/5R \qquad (12)$$

The coefficient 0.000627 is evaluated on the basis of a radius of 1.45×10^{-13}cm. This semi-empirical formula does not take into account changes in binding energy with shell structure, which would modify the third term somewhat. The semi-empirical formula gives a smooth version of the variation of mass with A and Z, but it gives somewhat low values for light masses and high values for heavy atoms. If we recalculate the coulomb energy on the basis of a radius 1.2×10^{-13} cm., the 6th term has to be modified to $0.00077\,Z^2/A^{1/3}$. We find that, if the 4th term is correspondingly modified to $0.013\,A^{2/3}$, fairly accurate values of masses can be calculated. A slightly better formula takes the 4th term as $0.0135\,A^{2/3}\,[1-(A/2-Z)/A]$.*

It is to be noted that the radius entering into these calculations is the proton distribution radius rather than the nuclear radius. The results in the preceding paragraphs show that an electromagnetic radius smaller than the customary 1.5×10^{-13} cm. gives much better agreement between theory and experiment. This cannot be taken as a confirmation of the value of 1.2×10^{-13} cm. for such a radius. It lends some weight, however, to the interpretation of the experiments by Fitch and Rainwater that the nuclear radius is smaller than previously assumed. A radius of 1×10^{-13} cm. would give still better agreement.

II. NON-UNIFORM CHARGE DISTRIBUTION

Experiments of high energy scattering by Hofstadter et al.[16] have indicated that the charge distribution in the nucleus is not uniform. Their results can be interpreted by means of a distribution which is peaked at the centre and falls off slowly towards the edge. Either an exponential or a gaussian charge distribution can be fitted to the results. The root mean square radius of the best fit is, on the average, smaller by about 25% than that calculated on a basis of $1.5\times10^{-13}A^{1/3}$ cm.

A non-uniform charge distribution changes the simple picture of the collective model considerably. Since the charge distribution has not yet been measured satisfactorily, a detailed calculation of such a collective model is premature. Some qualitative observations can be made, however.

Schiff[17] has pointed out that any experiment which measures the average value of the difference between the actual potential and that of an equal point charge measures, in reality, the root mean square radius of such a charge distribution. This can be illustrated in the calculations of the isotope shift. For any given charge distribution σ the

* Green and Engler[28] find that the coulomb energy term is best given by $0.00075\,Z^2/A^{1/3}$

equivalent radius R_{equ} of a uniform charge distribution which would give the same isotope shift is given by

$$\int_0^\infty \sigma R^4 dR = 3/5(Ze/4\pi)R^2_{equ} \qquad (13)$$

Therefore, if the charge distribution is known the equivalent radius can be found, and the isotope shift can be calculated by replacing R in equation (10) by R_{equ}. Conversely, if the theory of the isotope shift is to be trusted, the equivalent radius can be calculated from the measured isotope shift and can be compared with other experimental results which measure the effective radius or give an indication of the charge distribution.

This can be done, for example, for lead and gold, for which data from high energy scattering are available. Hofstadter et al.[16,17] find their results can be fitted to an exponential, modified exponential ($\sigma=\sigma_0[1+R/a]e^{-R/a}$), or Gaussian charge distribution with r.m.s. radii of 5.5—10.0, 4.7—6.8, and 3.6—4.5×10^{-13} cm. respectively. The best fit is at 7.95, 5.5, and 4.3 respectively. The r.m.s. radius for a uniform charge distribution using $1.5\times10^{-13}A^{1/3}$ is 6.8×10^{-13} for lead.

The effective radius as calculated from isotope shift is $5.3\pm2\times10^{-13}$ cm. These results should be compared as well with the effective radius obtained from μ mesonic X-ray measurememts for lead, which give a radius of 6.9×10^{-13}. It is seen that these results seem to agree fairly well for exponential distributions but not quite so well for the Gaussian distribution.

A potential similar to that of an oscillator potential, but with rounded corners, probably would not greatly change the level order as given by a simple oscillator potential. Such a potential, however, would give considerably better results in calculations of first excited states of even-even nuclei and quadrupole moments. The effective mass taking part in the collective motion which can be distorted would, in such a distribution, be smaller and a function of the shape of the potential. The energy difference of the excited states would be larger and, therefore, fit the experimental results much better without assuming unreasonable nuclear deformations. The quadrupole moments would be smaller not only because of the smaller effective radius but also due to the smaller resulting deformation. This would be only slightly offset by the increase in coulomb energy.

APPENDIX

(A)

A detailed formulation of the collective model of the nucleus will be found in a paper by Bohr and Mottelson[10]. Here only the necessary equations used in the calculations are brought down in a condensed form.

The nuclear Hamiltonian can be written as

$$H = H_p + H_s + H_{int} \qquad (14)$$

where H_p is the particle Hamiltonian with particle coordinates r,σ with respect to the nuclear axes, H_s is the surface Hamiltonian with distortion coordinates β,γ, relative to the nuclear axes, and H_{int} is the interaction energy. The nuclear axes are defined with respect to the space axes by means of Euler angles.

The surface Hamiltonian consists of

$$H_s = T_{vib} + T_{rot} + V_s \qquad (15)$$

The vibrational kinetic energy is

$$T_{vib} = -\frac{h^2}{8\pi^2 B}\left(\frac{1}{\beta^4}\frac{\partial\beta^4}{\partial\beta}\frac{\partial}{\partial\beta} + \frac{1}{\beta^2}\frac{1}{\sin 3\gamma}\frac{\partial \sin 3\gamma}{\partial\gamma}\frac{\partial}{\partial\gamma}\right) \qquad (16)$$

The rotational energy is

$$T_{rot} = h^2(I_k - j_k)/8\pi^2 j_k \qquad (17)$$

where I_k and j_k are the components of the total angular momentum and of the particle along the nuclear axes respectively.

The moment of inertia $j_k = 4B\beta^2\sin^2[\gamma - (2\pi/3)x] \qquad x = 1,2,3 \qquad (18)$

The surface potential energy is given by

$$V_s = (1/2)C\beta^2 \qquad (19)$$

The constants C and B represent the restoring force (surface tension minus coulomb force) and the effective mass taking part in the collective motion given by equations (2) and (3) respectively.

The interaction matrix elements of importance in these calculations, averaged during the rapid motion of the particle, are given by[26]

$$H_{int} = \langle mj \mid H_{int} \mid mj \rangle = k_{nl}(5/4\pi)^{1/2}(\beta\cos j)[3m^2_j - j(j+1)]/[4j(j+1)] \qquad (20)$$
$$k_{nl} = V_0 R^3 \mid R_{nl}(R) \mid^2$$

where V_0 is the depth of the potential well and R_{nl} is the normalized radial wave function.

For the ground state, the matrix element is given by

$$(5/4\pi)^{1/2}k_{nl}\ (\beta\cos j)\ (2j-1)/[4(j+1)].$$

for levels near the bottom of the well and for $x_0 = (2MV)^{1/2}2\pi R/h \gg 1$. Feenberg and Hammack[26] show that this expression can be approximated by $2T_{nl} = [x_0/(1+x_0)]^2$ where $2T_{nl} = h^2 w_{nl}/4\pi MR^2$ is the kinetic energy for an infinite depth and w_{nl} the n th zero of the Bessel function of order $1+1/2$.

Assuming $V_0 = 28$ Mev and $R = 1.5\times10^{-13}$ cm, they find that $H_{int} = 40$ Mev and $T_{14}[(x_0/(1+x_0)]^2$ about 25 Mev. A radius of $R = 1.2\times10^{-13}$ cm would give for a potential well of about 40 Mev, $T_{14}[x_0/(1+x_0)]^2$ about 35 Mev and H_{int} about 54 Mev.

It is convenient to define an interaction parameter (Eq. 4)

$$x = (5/16)^{1/2}k(\hbar wC/2\pi)^{-1/2}/j \qquad (4)$$

x and β are related by

$$\beta = [(5/4\pi)^{1/2}\,k/C][(2j-1)/4(j+1)] = [xj^{1/2}\,(2j-1)/2(j+1)][hw/2\pi C] \qquad (21)$$

x and x' refer, in the calculations of the quadrupole moments, to a coupling constant of k 40 Mev and x'' to an interaction constant of 54 Mev (i.e. $x'' \sim 1.3x'$), both assumed to be independent of n and l.

(B)

The quadrupole moment according to the collective model is given by two parts, as in equation 5

$$Q_L = Q_{sp} + Q_s \qquad (5)$$

If we limit ourselves to one particle outside a closed shell, the calculations are fairly simple

$$Q_{sp} = \langle m=j \mid r^2(3\cos^2\theta-1) \mid m=j \rangle = -[(2j-1)/(2j+1)]\overline{r^2} \quad (26)$$

r^2 is for usual potential wells of the order of $3/5R^2$. For a single neutron outside the closed shell, the QM is smaller by a factor of Z/A.

The surface quadrupole moment is given by

$$Q_s = Q_0 P_Q(x) \quad (6)$$

where the intrinsic quadrupole moment Q_0 is defined as in (7) and (8).

For strong coupling the factor P_Q is given as

$$P_Q = [I/(I+1)]\ [(2I-1)(2I+3)] \quad (27)$$

In the weak coupling approximation one finds by perturbation theory for $I = j \gg 1$

$$P_Q(x) = 1 - 3[(2I+1)/(I+1)\ (2I+3)]\ [x^2/(x^4+4/9)^{1/2}] \quad (28)$$

By means of formulas (5), (6), (7), 8), (27), and (28) the various quadrupole moments are calculated.

(C)

The total energy for excited states of even-even nuclei is

$$E = E_s + E_{int} + E_{rot} + E_{vib} \quad (29)$$

E_{vib} is not very much influenced by a rotational transition, and E_s and E_{int} are nearly the same for the ground and first excited state. The rotational energy to a first approximation is given by

$$E_{rot} = [h^2/8\ \pi^2 J][I(I+1)]$$

The moment of inertia j is given by

$$j = 3B\beta_E^2,$$

For $I = 2$ this reduces to

$$E_{rot} = h^2/4\ \pi^2 B\beta_E^2$$

or

$$\mid \beta_E \mid = (h/2\ \pi)\ (E_{rot}\ B)^{-1/2} \quad (30)$$

The quadrupole moment is, however, given by equations (7), (8), (27), and (28) and therefore

$$\mid \beta_Q \mid = (5\ \pi)^{1/2}(1/3ZR^2)[(I+1)(2I+3)/I(2I-1)]Q_{obs} \quad (31)$$

It is by means of equations (30) and (31) that β_E and β_Q are compared, and it is found that for $R = 1.5\times10^{-13}$ cm the ratio of β_E/β_Q is approximately 2.

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ON THE RADIATION INDUCED ACCELERATION OF ELEMENTARY ELECTRIC PARTICLES

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I. INTRODUCTION

Elementary electric particle accelerators developed until now to a high degree of perfection, are based upon the ponderomotive effect of electromagnetic waves, whose electric field components are parallel to the velocity vector of the particle. In order to transmit energy in the same sense at all times, it is necessary that coherence of the resulting Coulomb force and the particle be maintained during the whole process of acceleration. This imposes the condition of equality at any time between the phase velocity of the driving wave and the particle velocity. This basic working condition, designated as synchronism, can be satisfied either dynamically by variations of the electromagnetic field with time, or kinematically by suitably changing the length of the accelerating chambers through which the particle passes successively. In several accelerators both methods are used in varying combinations to achieve a synchronous guiding mechanism.

The above mentioned accelerators use the Coulomb force, which is parallel to the electric field vector of the driving wave. But it is well known from physical principles that in a radiation field another force acts on the particle at the same time, which is perpendicular to the Coulomb force and parallel to the wave normal. This force is the Lorentz force caused by the oscillations of the accelerated particle in the driving wave and manifests itself as radiation pressure.

No technical use has been made up to the present of this radiation pressure for particle acceleration. This may be due to the fact that the absolute value of the Lorentz force, which originates from a second order effect, is always small compared with the Coulomb force acting on the particle at the same time. Notwithstanding its small magnitude, the use of the radiation pressure of the propagated electromagnetic wave for particle acceleration has an important advantage over the use of the Coulomb force: without any external synchronizing means it always acts parallel to and in the same sense as the Poynting vector. The question therefore arises, whether the Lorentz force cannot be increased to the order of magnitude of the Coulomb force. This investigation will show that this is possible, if the particle, accelerated by the radiation pressure, is subjected to the additional guiding force of a magnetic field parallel to the wave normal. Since combined electromagnetic fields of this kind may occur in the vicinity of sunspots and similar stellar phenomena, the following analysis might be applicable to the genetics of cosmic rays. The same concepts can be applied to the development of a "radiation cyclotron" which used in conjunction with existing particle accelerators might enlarge their scope.

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II. RADIATION PRESSURE ON STATIONARY CHARGE CARRIERS

The investigations will begin with the calculation of the force of the electromagnetic field of a plane, monochromatic wave of circular frequency ω, acting on a point-like ion with an invariant charge e and the restmass m_0.

Using the unprimed reference system of cartesian coordinates x, y, z and of time t it is assumed that the "primary wave" is propagated along the x-axis. The primary electrical field stress $\vec{E}$ (amplitude $\vec{E}_{max}$) is parallel to the y-axis and the primary magnetic field strength $\vec{H}$ (amplitude $\vec{H}_{max}$) is parallel to the z-axis. By suitable choice of the origin of time the wave is represented by the expressions

$$\vec{E} = \vec{E}_{max} \sin \omega[t-(x/c)] \tag{1}$$

$$\vec{H} = \vec{H}_{max} \sin \omega [t-(x/c)] \tag{2}$$

(c being the velocity of propagation of light in vacuum). $\vec{E}$ and $\vec{H}$ are connected by the relation

$$\vec{H} = \vec{E}/\sqrt{\Pi/\Delta} \tag{3}$$

where Δ and Π are the so called dielectric constant and permeability of empty space. The radiation vector

$$\vec{S} = \vec{E} \cdot \vec{H} \tag{4}$$

is parallel to the x axis and its time average is

$$\bar{\vec{S}} = (1/2)\vec{E}_{max} \cdot \vec{H}_{max} \tag{5}$$

Introduce temporarily non-electromagnetic forces, which bind the ion to the plane $x = 0$, where the ions move without friction. The differential equation of the ion motion according to relativistic mechanics of point-mass is

$$(d/dt)m(dy/dt) = e\vec{E}_{(x=0)} = e\vec{E}_{max} \cdot \sin \omega t \tag{6}$$

Assuming further that the field intensities of the primary wave are so weak that the inert mass of the ion m can be replaced by its restmass m_0, eq. (6) changes with sufficient accuracy into

$$m_0(d^2y/dt^2) = e\vec{E}_{max} \sin \omega t \tag{7}$$

whose solution is the harmonic oscillation

$$y = -y_{max} \sin \omega t; \quad y_{max} = (e/m_0)(\vec{E}_{max}/\omega^2) \tag{8}$$

The corresponding velocity is

$$v_y = dy/dt = -(e/m_0)(\vec{E}_{max}/\omega)\cos \omega t \tag{9}$$

and the power $\dot{P}$ taken from the field by the ion

$$\dot{P} = v_y \vec{E}_{(x=o)} = -(e/m_0)(\vec{E}^2_{max}/\omega)\sin \omega t \cos \omega t \qquad (10)$$

hence its time average vanishes:

$$\bar{\dot{P}} = 0. \qquad (11)$$

This result is however incompatible with the electrodynamics of an oscillating ion' because the charge carrier, vibrating according to eq. (8), represents a Hertz oscillator. This produces a secondary, electromagnetic wave, polarized parallel to the y-axis and propagated spherically from the centre of the oscillator. The time average of the transmitted radiation power of the wave is

$$\bar{P}_{rad} = (8\pi/3)(e^2/4\pi\Delta m_0 c^2)^2 \bar{\vec{S}} \qquad (12)$$

The power in the stationary oscillating state has to be provided by the primary wave, therefore

$$\bar{P}_{rad} = \bar{\dot{P}} \qquad (13)$$

which contradicts eq. (11).

The difficulty can be overcome in this way: eq. (12) is rewritten with the help of eqs. (3), (5) and (8) as

$$\bar{P}_{rad} = {}^1/_2(\sqrt{\Delta/\Pi})(8\pi/3)(ey_{max}\omega^2/4\pi\Delta c^2)^2 \qquad (14)$$

This indicates that a force due to radiation damping, F_{rad}, has to be added to the electrical force $F_{el}=eE_{(x=0)}$ of eq. (6). It is assumed here that F_{rad} is defined by the relation

$$F_{rad} = \varkappa(d^2v_y/dt^2) = \varkappa(d^3y/dt^3) \qquad (15)$$

where $\varkappa$ is a real, positive constant whose value is yet to be determined.

In order to prove the correctness of this assumption the instantaneous power extracted from the primary wave is computed:

$$-v_y F_{rad} = -\varkappa v_y (d^2v_y/dt^2) \qquad (16)$$

Its time average is

$$(\varkappa/2\pi)\int_{-\pi}^{+\pi}(v_y d^2v_y/dt^2)d(\omega t) = -(\varkappa/2\pi)[|\omega v_y (dv_y/dt)|_{-\pi}^{+\pi} - \int_{-\pi}^{+\pi}(dv_y/dt)^2 d(\omega t)] \qquad (17)$$

The integral-free term vanishes due to the periodicity of y, hence

$$(\varkappa/2\pi)\int_{-\pi}^{+\pi}(dv_y/dt)^2 d(\omega t) = (\varkappa/2)(\omega^2 y_m)^2 \qquad (18)$$

If the value of $\varkappa$ is taken as

$$\varkappa = \sqrt{\Delta/\Pi}(8\pi/3)(e/4\pi\Delta)^2(1/c^4) \tag{19}$$

the expression (18) represents exactly the radiation power of eq. (14).

The energy relations of the controlled ion are thus correct, if the differential eq. (7) is replaced by

$$m_0(d^2y/dt^2) = F_{el} + F_{rad} = e\vec{E}_{max} \sin\omega t + \varkappa(d^3y/dt^3) \tag{20}$$

Its quasi-stationary solution is

$$y = -(e/m_0)(\vec{E}_{max}/\omega^2)[1/\sqrt{1+(\omega\varkappa/m_0)^2}]\sin(\omega t+\psi);\quad \tan\psi = \omega\varkappa/m_0 \tag{21}$$

Defining now from eq. (12) the radius a of the cross section of the controlled ion by

$$\pi a^2 = \bar{P}_{rad}/\bar{S} = (8\pi/3)(e^2/4\pi\Delta m_0c^2)^2 = \sqrt{\Pi/\Delta}\ (e/m_0)^2\varkappa \tag{22}$$

and replacing the circular frequency ω of the primary wave by its wavelength λ, where

$$\lambda = 2\pi(c/\omega) \tag{23}$$

it follows that

$$\tan\psi = (\pi/2)\sqrt{8/3}\,(a/\lambda) \tag{24}$$

If e is the charge of an electron and m_0 its rest mass, then a is of the order of magnitude of 10^{-13} cm, which corresponds to the radius of Born's model of the electron. In the range of the spectrum of electromagnetic waves, whose wavelengths are those of γ-rays, or larger

$$\tan\psi \ll 1 \tag{25}$$

In addition to the Coulomb force $F_{el}=e\vec{E}_{(x=0)}$, which is always in the plane $x=0$, the Lorentz force F_L, acting on the moving particle, has to be considered. This force is proportional to the vector product of the particle velocity v and the magnetic induction $B=\Pi H$ and is parallel to the radiation vector $\vec{S}$. With the notation F_{magn} for the remaining component of the Lorentz force in the x direction and with eq. (3)

$$F_{magn} = ev_y\vec{B}_{(x=o)} = ev_y\Pi\vec{H}_{(x=o)} = (e/c)v_y\vec{E}_{(x=o)} \tag{26}$$

The time average of this force is obtained from eq. (1), (21), (22)

$$\bar{F}_{magn} = (1/2)(e^2/m_0)(\vec{E}^2_{max}/\omega c)[\sin\psi/\sqrt{1+(\omega\varkappa/m_0)^2}] = (\bar{\vec{S}}/c)\left\{\pi a^2/[1+(\omega\varkappa/m_0)^2]\right\} \tag{27}$$

and the radiation pressure on the controlled charge carrier can be defined by

$$p = (\bar{\vec{S}}/c)\left\{1/[1+(\omega\varkappa/m_0)^2]\right\} \tag{28}$$

This equation reduces in the case of relation (25) to

$$p_0 = \lim_{(\omega\varkappa/m_0)\to 0} p = \bar{\vec{S}}/c \tag{29}$$

The order of magnitude of the force F_{magn} can be gauged, if its action is replaced by that of a virtual, time independent, longitudinal electrical intensity E_{long} on the charge e at rest:

$$E_{long} = \overline{F}_{magn}''/e = (\pi a^2/e)(\overline{\vec{S}}/c)\left\{1/[1+(\omega x/m_0)^2]\right\} \tag{30}$$

The value of E_{long} for ions carrying one electron charge is in the case of relation (25)

$$E_{long} = 1.4 \cdot 10^{-12}\overline{\vec{S}} \text{ volt/cm} \tag{31}$$

if S is measured in watt/cm². In technical applications the magnitude of E_{long} is very small in comparison with the usual intensities of electrical accelerating fields; e.g. in radar apparatus, whose very strong radiation fields in cylindrical wave guides can reach values of $100 \cdot 10^6$ watt/cm²

$$E_{long} = 0.14 \text{ mV/cm} \tag{32}$$

On the other hand voltages of large magnitude can result due to E_{long} along astronomical paths. Expressing eq. (31) in units of light-years ($31.1 \times 10^6 \cdot 3 \cdot 10^{10} = 0.933 \cdot 10^{18}$ cm)

$$E_{long} = 1.31\overline{\vec{S}} \text{ MV/light-year} \tag{33}$$

Thus ions emitted from giant stars can obtain very large energies from the cosmic radiation field.

III. RADIATION PRESSURE ON MOVING CHARGE CARRIERS

If the charge carrier becomes free to leave the plane $x=0$, it will start to move due to the radiation pressure in the direction of the radiation vector. The influence of this motion on the forces acting on the ion will now be investigated.

For this purpose a new system of reference is introduced and the electromagnetic field is observed relative to the system of "primed" coordinates x', y', z' and time t', which remain constantly bound to the moving charge carrier. The old and the new system coincide at the time $t=t'=0$ and their respective coordinate axes of same designation remain always parallel. If at time t v_x is the velocity of the controlled ion relative to the "unprimed" system, then world events registered in the unprimed system are represented in the primed system with the help of a Lorentz transformation of the velocity v_x. As in the primed system the controlled charge carrier is at rest, the Newtonian force F'_{magn} acting on the particle is obtained from eq. (27) by replacing the radiation vector S by S' and the circular frequency ω by ω'.

According to the rules of Lorentz transformation it follows from eq. (1), (2), (3) and with

$$\beta = v_x/c \tag{34}$$

$$\vec{E}' \quad (\sqrt{\Pi/\Delta})\,\vec{H}' = [\sqrt{(1-\beta)/(1+\beta)}]\;\vec{E}_{max}\sin\omega'[t'-(x'/c)] \tag{35}$$

Due to the condition of phase invariance

$$\omega'[t'-(x/c)] = \omega[t-(x/c)] \tag{36}$$

hence

$$\omega' = [\sqrt{(1-\beta)/(1+\beta)}]\;\omega \tag{37}$$

Formally eq. (35) leads to the transformation of the Poynting vector

$$\vec{S}' = [(1-\beta)/(1+\beta)]\vec{S} \tag{38}$$

The physical meaning of this relation can be obtained by the transition from the wave representation of electrodynamics to the photon representation. The energy of a photon W in the unprimed system is with the help of Planck's quantum of action

$$W = (h/2\pi)\omega \tag{39}$$

and this decreases in the primed system to

$$W' = (h/2\pi)\omega' = W\sqrt{(1-\beta)/(1+\beta)} \tag{40}$$

The photons, which move with the velocity of light through space, have a concentration n in the unprimed system defined by

$$n = \vec{S}/cW \tag{41}$$

The concentration n' of the photon current relative to the primed system is obtained by the following consideration: the photon should be temporarily replaced by individual, virtual particles, which relative to the primed system move parallel to the positive x'-axis with equal velocities

$$dx'/dt' = w' < c \tag{42}$$

An observer travelling with these quasi-photons, which are at rest relative to him, would measure their eigen-concentration n_0. Their concentration n' in the primed system is due to the Lorentz contraction of a measuring rod rigidly connected with the quasi-photons

$$n' = n_0/\sqrt{1-(w'^2/c^2)} \tag{43}$$

The velocity w of the quasi-photons parallel to the x-axis of the unprimed system is obtained with the help of Einstein's velocity addition theorem

$$w = (w' + v_x)/[1 + (w' v_x/c^2)] \tag{44}$$

The quasi-photon concentration relative to the unprimed system is then

$$n = n_0/\sqrt{1-(w^2/c^2)} \tag{45}$$

Returning now from the quasi-photons to light quanta by going to the limit $w' \to c$, the following rule for the transformation of the photon concentration is obtained:

$$\begin{aligned}(n'/n) &= \lim_{w' \to c} \sqrt{[(1-(w^2/c^2)]/[1-(w'^2/c^2)]} = \\ &= \lim_{w' \to c} \sqrt{[1+(w/c)]/[1+(w'/c)]}\ \sqrt{[1-(w/c)]/[1-(w'/c)]} = \\ &= \lim_{w'=c} \sqrt{[(1+w/c)]/[1+(w'/c)]}\ \sqrt{[1-(v_x/c)]/[1+(w' v_x/c^2)]} = \sqrt{(1-\beta)/(1+\beta)}\end{aligned} \tag{46}$$

With the help of eq. (23) and (37) this relation can be rewritten

$$n\lambda = n'\lambda' = \text{const} \tag{47}$$

If the "length" of an individual photon is defined as the reciprocal of the photon concentration, the number of wavelengths contained in this "length" is invariant. Eq. (47) then physically represents a quantum of light.

According to eq. (41)

$$n' = \overline{\vec{S}'}/cW' \tag{48}$$

hence with reference to eq. (46) and (40)

$$\overline{\vec{S}'} = cW'n' = cWn\,(1-\beta)/(1+\beta) \quad = \quad \overline{\vec{S}}\,(1-\beta)/(1+\beta) \tag{49}$$

which confirms the relation (38). The transformation of eq. (27) into the primed reference system leads to

$$\overline{F'}_{magn} = (\overline{\vec{S}'}/c)\,\pi a^2/[1+(\omega' x/m_0)^2] = \tag{50}$$

$$= (\overline{\vec{S}}/c)[(1-\beta)/(1+\beta)]\,\pi a^2/\{\,1+(\omega x/m_0)^2[(1-\beta)/(1+\beta)]\,\}$$

According to the fundamental laws of relativistic mechanics the Newtonian force component parallel to the x axis in the primed system equals that in the unprimed system and also the same magnitude of the cross sectional radius a is observed perpendicularly to the direction of motion in both systems. Hence

$$p = \overline{F'}_{magn}/\pi a^2 = (\overline{\vec{S}}/c)(1-\beta)/[(1+\beta)+(\omega x/m_0)^2(1-\beta)] \tag{51}$$

defines the radiation pressure on the moving charge carrier, which with the condition (25) reduces to

$$p_0 = \lim_{\omega x/m_0 \to 0} p = (\overline{\vec{S}}/c)[(1-\beta)/(1+\beta)] \tag{52}$$

IV. MOTION OF A CHARGE CARRIER IN A HOMOGENEOUS RADIATION FIELD

With the assumption of a homogeneous field of radiation of the primary intensity S, whose frequency lies in the range of relation (25), the differential equation for longitudinal motion of the charge carrier parallel to the wave is

$$(d/dt)[m_0 c\beta/\sqrt{(1-\beta^2)}] = (\overline{\vec{S}}/c)\,\pi a^2[(1-\beta)/(1+\beta)] \tag{53}$$

The equation should be integrated with the initial conditions

$$x = 0; \quad \beta = (1/c)(dx/dt) = 0 \quad \text{for } t = 0 \tag{54}$$

For this purpose use is made of the substitution $\beta = \tanh u$ and of the definition

$$T = m_0 c^2/\overline{\vec{S}}\,\pi a^2 \tag{55}$$

T being the time constant of the acceleration process. Using "non-dimensional time units" $\tau = t/T$ the integration of eq. (53) results in

$$\tau = \int_0^{\operatorname{arctanh}\beta} e^{2u} \cosh u\,du = (1/2)\left((1/3)\left\{\,[(1+\beta)/(1-\beta)]^{3/2}-1\,\right\}+\left\{\,[(1+\beta)/(1-\beta)]^{1/2}-1\,\right\}\right) \tag{56}$$

A second integration gives the "non-dimensional length" $\xi = x/cT$

$$\int_0^{\tau} \beta\,d\tau = \int_0^{\operatorname{arctanh}\beta} e^{2u} \sinh u\,du = (1/2)\left((1/3)\left\{\,[(1+\beta)/(1-\beta)]^{3/2}-1\,\right\}-\left\{\,[(1+\beta)/(1-\beta)]^{1/2}-1\,\right\}\right) \tag{57}$$

Eqs. (56) and (57) are the parametric representation of the longitudinal motion of the charge carriers, which are accelerated by radiation. The nature of this motion is shown by Figure 2.

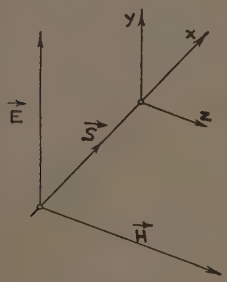

Figure 1
The radiation field

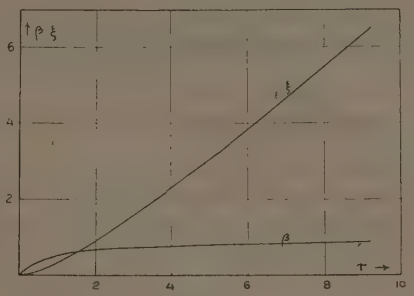

Figure 2
Acceleration of a charge carrier by a homogeneous field of radiation

V. RADIATION INDUCED ACCELERATION IN A HOMOGENEOUS, MAGNETIC FIELD

A homogeneous, time independent magnetic field is superposed on the primary electromagnetic wave. The induction vector of this field B is parallel to the primary radiation vector $\vec{S}$. The cartesian components of the electrical field in the primed system are then

$$E'_x = 0; \quad E'_y = \sqrt{(1-\beta)/(1+\beta)}\,[\vec{E}_{max} \sin \omega'(t'-x'/c)]; \quad E'_z = 0 \tag{58}$$

and those of the magnetic induction

$$B'_x = B; \quad B'_y = 0; \quad B'_z = \sqrt{(1-\beta)/(1+\beta)}\,[(\vec{E}_{max}/c) \sin \omega'(t'-x'/c)] \tag{59}$$

For brevity the treatment will be restricted to electrons ($e = -q_0$). The components of the Lorentz force acting on such a particle are:

$$F'_{magn,x} = -(q_0 v'_y/c)\,[\sqrt{(1-\beta)/(1+\beta)}\;\vec{E}_{max} \sin \omega'(t'-x'/c)]; \quad F'_{magn,y} = q_0 v'_z B;$$

$$F'_{magn,z} = -q_0 v'_y B \tag{60}$$

and the equations of motion of the controlled electron are

$$(d/dt')(m'dx'/dt) = -q_0(dy'/dt')(1/c)\sqrt{(1-\beta)/(1+\beta)}\;\vec{E}_{max} \sin \omega't' + \varkappa(d^3x'/dt'^3) \tag{61}$$

$$(d/dt')(m'dy'/dt') = -q_0[\sqrt{(1-\beta)/(1+\beta)}\;\vec{E}_{max} \sin \omega't' - (dz'/dt')B] + \varkappa(d^3y'/dt'^3) \tag{62}$$

$$(d/dt')(m'dz'/dt') = -q_0(dy'/dt')B + \varkappa(d^3z'/dt'^3) \tag{63}$$

By definition the equations are valid only as long as x' is not noticeably different from zero. This is the case for a short time interval. Considering then the y' and z' components of the motion, it can be assumed that the resulting velocity $v' = \sqrt{v_y'^2 + v_z'^2}$ is so small relative to c that it is permissible to replace m' in the primed system by its restmass m_0. For simpler notation the variables

$$\omega_0 = (q_0/m_0)B; \quad R = (\vec{E}_{max}/\omega_0 B)\sqrt{(1-\beta)/(1+\beta)} = (m_0/q_0)(\vec{E}_{max}/B^2)\sqrt{(1-\beta)/(1+\beta)} \quad (64)$$

will be used and the coordinates y', z', t' will be replaced by $\eta = y'/R$, $\zeta = z'/R$, $\tau = \omega_0 t'$ resp. Eqs. (62), (63) reduce then to

$$(d^2\eta/d\tau^2) - (d\zeta/d\tau) - (\omega_0 \varkappa/m_0)(d^3\eta/d\tau^3) = (i/2)(e^{i(\omega'/\omega_0)\tau} - e^{-i(\omega'/\omega_0)\tau}) \quad (65)$$

$$(d^2\zeta/d\tau^2) + (d\eta/d\tau) - (\omega_0 \varkappa/m_0)(d^3\zeta/d\tau^3) = 0 \quad (66)$$

Solution is obtained with

$$\eta = C_+ e^{i(\omega'/\omega_0)\tau} + C_- e^{-i(\omega'/\omega_0)\tau}; \quad \zeta = D_+ e^{i(\omega'/\omega_0)\tau} + D_- e^{-i(\omega'/\omega_0)\tau} \quad (67)$$

Four linear equations are obtained for the constants C_+, C_-, D_+, D_-,

$$-C_\pm (\omega'/\omega_0)^2[1 \mp i(\omega_0 \varkappa/m_0)(\omega'/\omega_0)] \mp D_\pm i(\omega'/\omega_0) = \pm i/2 \quad (68)$$

$$\pm C_\pm i(\omega/\omega_0) - D_\pm (\omega'/\omega_0)^2[1 \mp i(\omega_0 \varkappa/m_0)(\omega'/\omega)] = 0 \quad (69)$$

from which

$$C_\pm = \pm (i/2)[1 \mp i(\omega_0 \varkappa/m_0)(\omega'/\omega_0)]/\{1-(\omega'/\omega_0)^2[1 \mp i(\omega_0 \varkappa/m_0)(\omega'/\omega_0)]^2\} \quad (70)$$

$$D_\pm = -(1/2)(\omega_0/\omega)1/\{1-(\omega'/\omega_0)^2[1 \mp i(\omega_0 \varkappa/m_0)(\omega'/\omega_0)]^2\} \quad (71)$$

Using the expression

$$N = \{1-(\omega'/\omega_0)^2[1-(\omega_0 \varkappa/m_0)^2(\omega'/\omega_0)^2]\}^2 + [2(\omega_0 \varkappa/m_0)(\omega'/\omega_0)^3]^2 \quad (72)$$

the constants are

$$C_\pm = \pm (i/2N)\{1-(\omega'/\omega_0)^2[1+(\omega_0 \varkappa/m_0)^2(\omega'/\omega_0)^2] \mp i(\omega_0 \varkappa/m_0)(\omega'/\omega_0) \cdot \left\{1+(\omega'/\omega_0)^2[1+(\omega_0 \varkappa/m_0)^2(\omega'/\omega_0)^2]\right\}\} \quad (73)$$

$$D_\pm = -(1/2N)(\omega_0/\omega)\{1-(\omega'/\omega_0)^2[1-(\omega_0 \varkappa/m_0)^2(\omega'/\omega_0)^2] \mp 2i(\omega_0 \varkappa/m_0)(\omega'/\omega_0)^3\} \quad (74)$$

Replacing these expressions into eq. (67)

$$\eta = -\{1-(\omega'/\omega_0)^2[1+(\omega_0 \varkappa/m_0)^2(\omega'/\omega_0)^2]\}[\sin(\omega'/\omega_0)\tau](1/N) + (\omega_0 \varkappa/m_0)(\omega'/\omega_0) \cdot \left\{1+(\omega'/\omega_0)^2[1+(\omega_0 \varkappa/m_0)^2(\omega'/\omega_0)^2]\right\}[\cos(\omega'/\omega_0)\tau](1/N) \quad (75)$$

$$\zeta = (-\omega_0/\omega')\{1-(\omega'/\omega_0)^2[1-(\omega_0 \varkappa/m_0)^2(\omega'/\omega_0)^2]\}[\cos(\omega'/\omega_0)\tau](1/N) - 2(\omega_0 \varkappa/m_0)(\omega'/\omega_0)^2[\sin(\omega'/\omega_0)\tau](1/N) \quad (76)$$

From eq. (75) follows

$$(dy'/dt') = \omega_0 R(d\eta/d\tau) = -(\omega'/\omega_0^2)(q_0/m_0)\vec{E}_{max}\sqrt{(1-\beta)/(1+\beta)}\left(1-(\omega'/\omega_0)\right)^2 [1+(\omega_0 \varkappa/m_0)^2(\omega'/\omega_0)^2](\cos\omega' t')(1/N) + (\omega_0 \varkappa/m_0)(\omega'/\omega_0)\{1+(\omega'/\omega_0)^2[1+(\omega_0 \varkappa/m_0)]^2 \cdot (\omega'/\omega_0)^2]\}(\sin\omega' t')(1/N) \quad (77)$$

hence the time average of the Lorentz force component parallel to the wave is

$$\overline{F}_{magn,x} = (\omega'/\omega_0^2)(q_0^2/m_0)(\vec{E}^2_{max}/2c)[(1-\beta)/(1+\beta)](\omega_0 \varkappa/m_0)(\omega'/\omega) \cdot \{1+(\omega'/\omega_0)^2[1+(\omega_0 \varkappa/m_0)^2(\omega'/\omega_0)^2]\}(1/N) \quad (78)$$

If the longitudinal magnetic field vanishes ($\omega_0 \to 0$), the relation reduces to

$$\overline{F'}^{(0)}_{magn,x} = \lim_{\omega_0 \to 0} \overline{F'}_{magn,x} = (q_0^2/m_0^2)(\overline{E^2}_{max}/2c)[(1-\beta)/(1+\beta)]\varkappa/[1+(\omega'\varkappa/m_0)^2] \quad (79)$$

which due to the relation (22) is identical with the expression (50) for $\overline{F'}_{magn}$. Under otherwise equal conditions the radiation pressure on the electron is increased by the action of the longitudinal magnetic field in the ratio

$$\overline{F'}_{magn,x}/\overline{F'}^{(0)}_{magn,x} = (\omega'/\omega_0)^2\left[1+(\omega'\varkappa/m_0)^2\right]\left[1+(\omega'/\omega_0)^2[1+(\omega'\varkappa/m_0)^2]\right](1/N) \quad (80)$$

Excluding again the case of rays with very short wavelengths, $\omega'\varkappa/m_0$ can be neglected compared with unity and eq. (80) becomes with sufficient accuracy

$$\overline{F'}_{magn,x}/\overline{F'}^{(0)}_{magn,x} = (\omega'/\omega_0)^2[1+(\omega'/\omega_0)^2]/\left\{[1-(\omega'/\omega_0)^2]^2+[2(\omega'\varkappa/m_0)(\omega'/\omega_0)^2]^2\right\} \quad (81)$$

This relation reduces in the case of resonance ($\omega' \to \omega_0$) to

$$\lim_{\omega' \to \omega_0}(\overline{F'}_{magn,x}/\overline{F'}^{(0)}_{magn,x}) = (1/2)(m_0/\omega_0\varkappa)^2 \quad (82)$$

As an example the case $B = 8 \cdot 10^{-6}$ Vs/cm^2; $\omega_0 = 2\pi f_0 = 1.41 \cdot 10^{10}$rad/sec; $f_0 = 225$ MHz; $\lambda_0 = 13.3$ cm will be calculated. With eq. (21) and (24) $\omega_0\varkappa/m_0 = 0.89 \cdot 10^{-13}$ and the radiation pressure at resonance is

$$1/2 \cdot 10^{26}/0.89^2 = 6.4 \cdot 10^{25}$$

times larger than in the case when only the wave is acting. If

$$\omega > \omega_0 \quad (83)$$

is chosen, during the acceleration process ω' approaches the resonance frequency ω_0 according to eq. (37) and becomes equal to it, when the longitudinal velocity $\beta = \beta_0$. This corresponds to the conditions

$$\omega_0 = \omega\sqrt{(1-\beta_0)/(1+\beta_0)}; \quad \beta_0 = (\omega^2-\omega_0^2)/(\omega^2+\omega_0^2) \quad (84)$$

The total translational energy of the electron is then

$$W_{el} = m_0c^2/\sqrt{1-\beta_0^2} = m_0c^2[(\omega/\omega_0)+(\omega_0/\omega)]/2 \quad (85)$$

and its kinetic energy is

$$T = W_{el} - m_0c^2 = (m_0c^2/2)(\sqrt{\omega/\omega_0} - \sqrt{\omega_0/\omega})^2 \quad (86)$$

By suitable choice of $\omega/\omega_0 > 1$, T can be made arbitrarily large.

Usually very long paths are needed to accelerate a charged elementary particle from rest into the resonance oscillations, only by the mechanism of radiation pressure. This however is not important, if the acceleration of elementary particles in the locally strongly magnetized ionosphere of radiating stars is considered in the light of the previously developed theory. It has to be realized only that here the particles are not accelerated by a monochromatic light wave, but by the pressure of a more or less continuous spectrum of incoherent electromagnetic oscillations. Hence the described theory has to be enlarged in this case by a statistical treatment. The application of the

above developed ideas under terrestrial conditions to a particle accelerator requires a "preaccelerator" of a different kind, which would be used to reduce the exceedingly long acceleration path. It is not intended to discuss this basic possibility in detail here.

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ANGULAR MOMENTUM IN NON-SPHERICAL FIELDS

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INTRODUCTION

Experimental evidences on nuclear quadrupole moments and some other nuclear properties suggest that relatively big deviations from spherical symmetry are associated with the equilibrium state of some nuclei[1,2]. There is also good evidence for some regularities in these deformations[3], and it is most probable that some quantities related to nuclear asphericity constitute important parameters in the description of nuclear states and should be properly taken into account.

It is customary to approach nuclear problems by the method of self-consistent field. One then visualizes each nucleon as moving in a smooth field equivalent, in a certain sense and to a certain extent, to the actual field experienced by the nucleon.

The shape of the self-consistent field is closely connected with the shape of the distribution function of nuclear matter, and it is therefore of interest to study the motion of particles in fields which are not spherically symmetric. Such problems arise, of course, very often when treating the motion of particles in external fields; in the present case, however, the deformations of the field are not imposed from the outside but are connected with the presence of the particle in the field. The field should therefore be treated as a part of the system and its dynamics should be included in all dynamical considerations.

THE NON-SPHERICAL FIELD

The system under consideration will consist of a field plus a particle; its Hamiltonian is

$$H = H_p + H_f + V(\mathbf{r}) \qquad (1)$$

In this expression H_p is the Hamiltonian of the free particle, H_f — that of the free field, and $V(r)$ is the interaction of the particle with the field.

It is convenient to develop $V(\mathbf{r})$ in a series of spherical harmonics:

$$V(\mathbf{r}) = \sum b_{n\nu}(r)\, Y_{n\nu}(\theta,\varphi) + b^*_{n\nu}(r)\, Y^*_{n\nu}(\theta,\varphi) \qquad (2)$$

where we have chosen this form to exhibit the fact that $V(\mathbf{r})$ is a real function of its argument. Since, by hypothesis, our system is isolated, a rotation of the coordinates in space should leave $V(\mathbf{r})$ invariant. This implies a certain transformation-character for the b's, namely: $b_{n\nu}$ should transform like $Y^*_{n\nu}$, and $b^*_{n\nu}$ — like $Y_{n\nu}$. In other words: a rotation R in the 3-dimensional coordinate space induces a transformation $B(R)$ in the b-space, and the $2n+1$ quantities $b^*_{n\nu}$ form the components of an irreducible tensor of order n in this space.

One can define an operator J^f_μ ($\mu = -1,\ 0,\ +1$) operating in the b-space which would correspond to the angular momentum associated with the degrees of freedom b. This $\mathbf{J}^f$ should satisfy the following commutation relations:

$$[J^f_\mu, J^f_{\mu'}] = q_{\mu\mu'} J^f_{\mu+\mu'} \qquad (3)$$

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where $q_{\mu\mu}$, is an antisymmetric matrix with

$$q_{-1,0} = q_{0,1} = q_{-1,1} = 1 \tag{3}$$

(The tensor components of a general angular momentum vector are constructed from its ordinary x, y and z components according to the following definitions:

$$J_{-1} = (J_x - iJ_y)/\sqrt{2},\ J_0 = J_z,\ J_1 = -(J_x + iJ_y)/\sqrt{2}).$$

In addidion, since the b^*'s are tensor operators with respect to J^f, this vector will also satisfy the following commutation relations:

$$\begin{aligned} [J^f_\mu, b^*_{n\nu}\ ] &= g^n_{\mu,\nu}\ \ b_{n\mu+\nu} \\ [J^f_\mu, b_{n\nu}\ ] &= g^n_{\mu,-\nu}\ \ b_{n\nu-\mu}\,(-1)^\mu \end{aligned} \tag{4}$$

where the g's are given by[4]

$$g^n_{\mu,\nu} = \begin{cases} [(n-\nu+1)\,(n+\nu)/2]^{\frac{1}{2}} & \mu = -1 \\ \nu & \mu = 0 \\ -[(n+\nu+1)\ (n-\nu)/2]^{\frac{1}{2}} & \mu = +1 \end{cases} \tag{5}$$

As $b_{n\nu}(r)$ is a function of the magnitude of r only it commutes with the particle angular momentum operator $\mathbf{J}^P$. Noting that the spherical harmonics are tensor operators with respect to $\mathbf{J}^P$ (and therefore satisfy with $\mathbf{J}^P$ commutation relations similar to (4) with $\mathbf{J}^P$ replacing $\mathbf{J}^f$ and Y's replacing b^*'s) we find:

$$\Sigma_\nu [J^f_\mu + J^p_\mu,\ b_{n\nu} Y_{n\nu}] = \Sigma_\nu [(-1)^\mu g^n_{\mu,-\nu-\mu} + g^n_{\mu,\nu}]\ \ b_{n\nu}\, Y_{n\mu+\nu} = 0 \tag{6}$$

where the last step can be deduced from (5). We could anticipate this result since $\Sigma_\nu b_{n\nu} Y_{n\nu}$ is the scalar product of the two operators each of which is a tensor with respect to $\mathbf{J}^f + \mathbf{J}^P$.

Evidently $\mathbf{J}^P$ alone does not commute with the interaction (2); the particle's angular momentum is not a good quantum number in a non-spherical field since due to the reaction on the field angular momentum is shared with the field. Equation (6) says that one should add $\mathbf{J}^f$ to $\mathbf{J}^P$ in order to get a quantity which does commute with $V(\mathbf{r})$. $\mathbf{J}^f$ can therefore be justly called the field angular momentum.

THE CONSTRUCTION OF $\mathbf{J}^f$ FROM THE FIELD VARIABLES

A knowledge of H_f usually enables us to construct the appropriate angular momentum vector for the field. However, since the set of b's determines the field in our case we can try to take them as the generalized coordinates of the field and construct the field angular momentum from them. H_f will, of course, also be a function of the b's, and as it should be invariant to rotations of space it will automatically commute with $\mathbf{J}_f$.

It is known[5] that if T^{k_1} and T^{k_2} are both reduced tensor operators of degrees k_1 and k_2 respectively, then T^{k_3}, defined by:

$$T^{k_3}_{\varkappa_3} = \Sigma_{\varkappa_1\varkappa_2} (k_1 k_2 \varkappa_1 \varkappa_2 |\ k_1 k_2 k_3 \varkappa_3)\ T^{k_1}_{\varkappa_1} T^{k_2}_{\varkappa_2}$$

is also a reduced tensor operator of degree k_3 (the expression in brackets is the usual Clebsch-Gordan coefficient).

Since $\mathbf{J}^f$ has to be a tensor of the first degree — a vector — we can construct its most general expression from the b's by the same rule. Noting that $b^*_{n\nu}$ and $(-1)^\nu b_{n-\nu}$ are tensors of degree n we get as an expression for the field angular momentum:

$$J^f_\mu = \Sigma_{n\nu}(n_1 n_2 \nu_1 \nu_2 | n_1 n_2 1\mu)\,[\alpha_{n_1 n_2}\, b^*_{n_1 \nu_1} b^*_{n_2 \nu_2} + (-1)^{\nu_1+\nu_2}\,\alpha'_{n_1 n_2}\, b_{n_1-\nu_1} b_{n_2-\nu_2}$$
$$+\lambda_{n_1 n_2}(-1)^{n_1-\nu_1} b_{n_1-\nu_1} b^*_{n_2 \nu_2} + \lambda'_{n_1 n_2}(-1)^{n_2-\nu_2} b^*_{n_1 \nu_1} b_{n_2-\nu_2} \qquad (7)$$

where the α's and the λ's have still to be determined by making use of the relations (3) and (4) which are postulated for $\mathbf{J}^f$.

We shall require the reality conditions for the field angular momentum:

$$J^{f*}_\mu = (-1)^\mu J^f_{-\mu} \qquad (8)$$

but we shall not impose any reality condition on the b's since we do not have to assume that they represent any measurable quantity.

In making use of equation (4) we note that it can determine $\mathbf{J}^f$ only up to an additive quantity which commutes with the b's and b^*'s but is otherwise arbitrary. This freedom is limited, however, by the relations (3) which characterize a vector as an angular momentum vector. We shall therefore find it convenient to use the following lemma:

If $\mathbf{j}$ is known to be a sum of two commuting vectors: $\mathbf{j} = \mathbf{j}_1 + \mathbf{j}_2$, and if in addition both j and j_1 satisfy (3) (i.e. both are angular momenta), then either $\mathbf{j}_2 = 0$ or $\mathbf{j}_2$ also satisfies (3). The proof is obvious.

Proceeding with the considerations which will enable us to specialize the values of α and λ in (7), we note that since each operator $b_{n\nu}$ refers to a different degree of freedom in the field, the only commutators which may differ from zero are $[b_{n\nu}, b^*_{n\nu}]$. Taking this into account, and in view of the vanishing of the Clebsch-Gordan coefficient in (7) whenever $\nu_1 + \nu_2 - \mu \neq 0$ we see that:

the first term in (7) (proportional to α) contributes nothing to $[J^f_\mu, b^*_{n\nu}]$;

the second term (proportional to α') contributes something proportional to $b_{n-(\mu+\nu)}$;

and only the third and fourth terms contribute something proportional to $b^*_{n\mu+\nu}$;

Since by (4) $[J^f_\mu, b^*_{n\nu}]$ is proportional to $b^*_{n\mu+\nu}$ the values of α and α' should be so chosen as to make the corresponding commutator with both b and b^* equal to zero. From this follows that that part of $\mathbf{J}^f$ which is due to terms proportional to α and α' cannot be an angular momentum, and by the previous lemma should be identically zero if the remaining terms do possess the property (3).

We shall therefore neglect the α terms altogether and shall rewrite (7) in the following form:

$$J^f_\mu = \Sigma_{n,\nu}[(-1)^{n_2-\nu_2}\lambda_{n_2 n_1}(n_2 n_1 \nu_2 \nu_1 | n_2 n_1 1\mu) + (-1)^{n_2-\nu_2}(n_1 n_2 \nu_1 \nu_2 | n_1 n_2 1\mu)\lambda'_{n_1 n_2}]\, b^*_{n_1\nu_1} b_{n_2-\nu_2} +$$
$$+\Sigma_{n,\nu}(-1)^{n_2-\nu_2}\lambda_{n_2 n_1}(n_2 n_1 \nu_2 \nu_1 | n_2 n_1 1\mu)\,[b_{n_2-\nu_2}, b^*_{n_1\nu_1}] = \Sigma(-1)^{n_2-\nu_2}(n_1 n_2 \nu_1 \nu_2 | n_1 n_2 1\mu)\times$$
$$\times(\lambda'_{n_1 n_2} - (-1)^{n_1+n_2}\lambda_{n_2 n_1})\, b^*_{n_1\nu_1} b_{n_2-\nu_2} + \Sigma(nn-\nu\nu | nn10)\lambda_{nn}(-1)^{n-\nu}\,[b_{n\nu}, b^*_{n\nu}]\,\delta_{\mu,0} \qquad (9)$$

For $\mu \neq 0$ we thus get the following expression for J^f_μ :

$$J^f_\mu = \Sigma(-1)^{n_2-\nu_2} \gamma_{n_1 n_2}(n_1 n_2 \nu_1 \nu_2 \,|\, n_1 n_2 1\mu)\; b^*_{n_1\nu_1}\, b_{n_2-\nu_2} \qquad (10)$$

Inserting this expression in equation (4) we have:

$$[J^f_\mu, b^*_{n\nu}] = g^n_{\mu,\nu} b^*_{n\mu+\nu} = \Sigma_{n_1\nu_1}(-1)^{n+\nu} \gamma_{n_1 n}(n_1 n \nu_1 - \nu |\, n_1 n 1\mu)\; b^*_{n_1\nu_1}[b_{n\nu}, b^*_{n\nu}] \qquad (11)$$

Since the b's operate on different degrees of freedom of the field, an equation of the form $\Sigma \lambda_{n\nu} b_{n\nu} = 0$ implies $\lambda_{n\nu} = 0$. We can therefore deduce from equation (11) that $\gamma_{n_1 n_2} = \gamma_{n_1}\delta_{n_1 n_2}$, and since the g's are c numbers the same should be true of the commutator $[b_{n\nu}, b^*_{n\nu}]$. The last conclusion enables us to adopt (10) as the general expression for the field angular momentum (also for $\mu = 0$) if we can show that it satisfies (3).

Since the commutator of the b's is a c-number, say β, we can define new b's by $\tilde{b}_n = b_n/(\beta_n)^{\frac{1}{2}}$ and thus arrive at the relation $[\tilde{b}_{n\nu}, \tilde{b}^*_{n\nu}] = 1$. Since we are going to determine the γ's of eq. (10), we can absorb the β's in the γ's and without any loss of generality assume that the b's in (10) are our new $\tilde{b}$'s.

It remains to be shown that the field angular momentum as defined in (10) satisfies (3). Once this has been shown the omission of the different terms we have neglected in (7) will be justified. Straightforward calculation yields:

$$[J^f_\mu,\; J^f_{\mu'}] = \Sigma \gamma_n^2 b^*_{n\nu_1} b_{n\nu_2} \{(-1)^\mu (nn\nu'_1\nu_2 |\, nn1-\mu)(nn\nu_1\nu'_1 |\, nn1\mu') - (-1)^{\mu'}(nn\nu_1\nu'_2 \,|\, nn1\mu)(nn\nu_2'\nu_2 |\, nn1-\mu')\} \qquad (12)$$

The expression in curly brackets can be evaluated by taking matrix elements of the identity $[j_\mu, j_{\mu'}] = q_{\mu\mu'} j_{\mu+\mu'}$ where j_μ are the components of an ordinary angular momentum vector. One finds that:

$$\Sigma_{\nu'_1\nu'_2} (-1)^\mu (nn\nu'_1\nu_2 |\, nn1-\mu)(nn\nu_1\nu'_1 \,|\, nn1\mu') - (-1)^{\mu'}(nn\nu_1\nu'_2 |\, nn1\mu)(nn\nu_2\nu_2 |\, nn1-\mu') = q_{\mu\mu'}(-1)^{n-\nu_2}[3/n(n+1)(2n+1)]^{\frac{1}{2}}(nn\nu_1-\nu_2 |\, nn1\mu+\mu') \qquad (13)$$

Substituting (13) in (12) we obtain:

$$[J^f_\mu,\; J^f_{\mu'}] = q_{\mu\mu'}\Sigma\gamma_n[3/n(n+1)(2n+1)]^{\frac{1}{2}}(-1)^{n-\nu'}\gamma_n(nn\nu\nu' |\, nn1\mu+\mu')b^*_{n\nu} b_{n-\nu'}$$

This expression reduces to the desired expression (3) if we choose

$$\gamma_n = [n(n+1)(2n+1)/3]^{\frac{1}{2}}$$

We thus finally arrive at the following expression for the angular momentum of the field:

$$J^f_\mu = \Sigma(-1)^{n-\nu'}[n(n+1)(2n+1)/3]^{\frac{1}{2}}\; (nn\nu\nu' |\, nn1\mu) b^*_{n\nu}\, b_{n-\nu'} \qquad (14)$$

For the z-component J_o^f (14) reduces to

$$J_o^f = \Sigma_\nu b_{n\nu}^* b_{n\nu} \qquad (15)$$

The eigenvalues of $b_{n\nu}^* b_{n\nu}$ are known to be the positive integers, and due to (15) we can interpret $N_{n\nu} = b_{n\nu}^* b_{n\nu}$ as the number of quanta of field oscillations at the position r which are associated with an angular momentum n and a z-component ν.

The expression (14) was obtained by A. Bohr[6] by actual calculation of the angular momentum of an oscillating incompressible liquid drop.

A remark should be added on the apparent generality of eq. (14). Clearly our basic assumption is the identification of the coefficients b with the generalized coordinates of the free field. To justify this choice one has to know the Hamiltonian of the free field in some coordinates and verify that the b's are obtained by a canonical transformation. Our result is indepenedent of the Hamiltonian only in as far as it meets with this requirement.

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MULTIPLE POINTS OF PATHS OF BROWNIAN MOTION IN THE PLANE *

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Let $(\Omega, \mathcal{E}, Pr)$ be a probability space, i.e., $\Omega = \{\omega\}$ is a set of elements ω, $\mathcal{E} = \{E\}$ is a Borel field of subsets of Ω called "events", and Pr is a countably additive measure defined on $\mathcal{E}$ and satisfying $Pr(\Omega) = 1$. $Pr(E)$ is called the "probability" of the event E.

A (mathematical) *one-dimensional Brownian motion*[1,5] is a real-valued function $x(t, \omega)$ of the two variables t and ω, defined for all non-negative real numbers t, $0 \leqslant t < \infty$, and for all $\omega \in \Omega$, which has the following properties:

(B_1) $x(0;\ \omega) \equiv 0$;

(B_2) for any real numbers s, t with $0 \leqslant s < t < \infty$, the "increment" $x(t, \omega) - x(s, \omega)$ is $\mathcal{E}$-measurable in ω and has a normal distribution with mean 0 and variance $t-s$, i.e.,**

$$E_{x,s,t,\alpha} \equiv \{\omega \mid x(t,\omega) - x(s,\omega) < \alpha\} \in \mathcal{E} \tag{1}$$

and

$$Pr(E_{x,s,t,\alpha}) = \left(1/\sqrt{2\pi(t-s)}\right) \int_{-\infty}^{\alpha} e^{-u^2/2(t-s)}\, du \tag{2}$$

for every real number α.

(B_3) for any real numbers s_i, t_i $(i = 1, \ldots, m)$ with $0 \leqslant s_1 < t_1 \leqslant s_2 < t_2 \leqslant \ldots \leqslant s_m < t_m < \infty$, the increments $x(t_i, \omega) - x(s_i, \omega)$, $i = 1, \ldots, m$ are independent in the sense of probability theory, i. e.,

$$Pr\left(\bigcap_{i=1}^{m} E_{x,s_i,t_i,\alpha_i}\right) = \prod_{i=1}^{m} Pr\left(E_{x,s_i,t_i,\alpha_i}\right) \tag{3}$$

for any real α_i, $i = 1, \ldots, m$.

A 2-dimensional or *plane Brownian motion* is an ordered pair of two mutually independent one-dimensional Brownian motions, i.e., a pair of one-dimensional Brownian motions $x(t, \omega)$ and $y(t, \omega)$ with the property that

$$Pr(E_{x,s,t,\alpha} \cap E_{y,s',t',\alpha'}) = Pr(E_{x,s,t,\alpha}) \cdot Pr(E_{y,s',t',\alpha'}) \tag{4}$$

for any real numbers s, t, α, s', t', α', with $0 \leqslant s < t$, $0 \leqslant s' < t'$.

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** $'\omega \mid \ldots'$ denotes the set of ω having the properties following the vertical bar; and similarly $'z \mid \ldots'$ etc.

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If we consider $z(t, \omega) = [x(t, \omega), y(t, \omega)]$ as a point in a Euclidean plane then, for each fixed ω, $z(t, \omega)$ may be considered as a function of t, defined for $0 \leqslant t < \infty$, and assuming as values points (or vectors) in the plane.

It is easy to see that this definition of a plane Brownian motion is independent of the choice of the rectangular coordinate system; i.e., the motion is isotropic, it is invariant vis-à-vis rotations of the coordinate system.

It is further assumed[1] that the Borel field $\mathcal{E}$ is already extended in such a manner that the subset C of Ω consisting of all ω for which $x(t, \omega)$ is a continuous function of t for $0 \leqslant t < \infty$ is $\mathcal{E}$-measurable and satisfies $P(C) = 1$.

For any point z in the plane, for any $\omega \in \Omega$ and any real numbers a, b with $0 \leqslant a < b < \infty$, let us put

$$L_{a,b}(z'; \omega) = \{z' + z(t, \omega) \mid a \leqslant t \leqslant b\}, \tag{5}$$

$$L_a(z'; \omega) = \{z' + z(t, \omega) \mid a \leqslant t < \infty\}, \tag{6}$$

$$L(z'; \omega) = L_0(z'; \omega) = \{z' + z(t, \omega) \mid 0 \leqslant t < \infty\} \tag{7}$$

where the $+$ sign in the above formula (as well as $+$ and $-$ in similar context in the sequel) refers to vector addition in the plane. Furthermore, when $z' = 0$, i.e., coincides with the origin, we use the abbreviations

$$L_{a,b}(\omega) = L_{a,b}(0; \omega), \quad L_a(\omega) = L_a(0; \omega), \quad L(\omega) = L(0; \omega). \tag{8}$$

$L_{a,b}(z'; \omega)$ is called the (a, b) *path* of the plane Brownian motion starting from z', and $L(z'; \omega)$ is called the *path* of the plane Brownian motion starting from z'.

For almost all ω, $L_{a,b}(z'; \omega)$ is a continuous image of the finite closed interval $\{t \mid a \leqslant t \leqslant b\}$ and is hence a compact subset of the plane.

A point z_0 in the plane is called a *k-multiple point* or a *multiple point of multiplicity k*, $(k = 2, 3, \ldots)$ of $L_{a,b}(z'; \omega)$ [resp. of $L_a(z'; \omega)$] if there exist k real numbers $t_1, \ldots, t_k$ with $a \leqslant t_1 < \ldots < t_k \leqslant b$ [resp. $a \leqslant t_1 < \ldots < t_k < \infty$] for which $z_0 = z' + z(t_i; \omega)$, $i = 1, \ldots, k$. It is clear that z_0 is a k-multiple point of $L_{a,b}(z'; \omega)$ [resp. $L_a(z'; \omega)$] if and only if $z_0 - z'$ is a k-tuple point of $L_{a,b}(\omega)$ [resp. $L_a(\omega)$].

P. Lévy[4] proved that almost all paths $L(\omega)$ have double (= 2-multiple) points. In a previous paper[2], we proved that if one considers Brownian motion in higher dimensional space then Lévy's result remains valid in 3-space, but that in a space of higher dimension than 3 almost all paths are free from double points. The main purpose of the present note is to prove that almost all paths of Brownian motion in the plane have points of arbitrary high (finite) multiplicity. Our methods are, in part, similar to those previously used[2], but we have to rely much more heavily on considerations of a combinatorial nature. We state explicitly our main result:

THEOREM 1. *Let* M *be the set of all* ω *for which* $L(\omega)$ *contains* k*-multiple points for every* $k = 2, 3, \ldots$, *then* $\Pr(M) = 1$.

Since the proof is rather involved, we shall lead to the theorem through a sequence of Lemmas.

For any point z in the plane we denote by $|z|$ the distance of z from the origin. We denote by $p(\rho, \beta)$ the probability that $L_{0,1}(z'; \omega)$ have a point in common with the circle $|z| \leqslant \rho$ when z' is a point at distance β from the origin (because of the isotropy of the Brownian motion this probability is the same for all z' with $|z'| = \beta$). In other words: we put for every $\beta \geqq 0$ and $\rho > 0$

$$p(\rho,\beta) = Pr\left(\inf_{0\leqq t\leqq 1} |(\beta,0)+z(t,\omega)| \leqslant \rho\right). \tag{9}$$

When $\beta = 1$ we abbreviate the notation and put

$$p(\rho) = p(\rho, 1). \tag{10}$$

LEMMA 1. p (ρ, β) *is a monotone non-decreasing function of* ρ *and a monotone non-increasing function of* β.

Proof. The assertion about ρ is obvious. To prove rigorously the other assertion, we use the homogeneity property of the Brownian motion (cf. e.g.[5]). According to this property, a space-scale change of $1 : \lambda$ is exactly compensated by a time-scale change of $1 : \lambda^2$; thus for every $\lambda > 0$ we have

$$p(\rho,\beta) = Pr\left(\inf_{0\leqq t\leqq \lambda^2} |(\lambda\beta, 0) + z(t,\omega)| \leqslant \lambda\rho\right),$$

whence it follows for $\lambda \geqq 1$ that

$$p(\rho, \beta) \geqq p(\lambda\rho, \lambda\beta) \geqq p(\rho, \lambda\beta).$$

LEMMA 2. *Let* $0 < \rho < \beta < R$, *then*

$$Pr(|(\beta, 0) + z(t, \omega)| \leqslant \rho \text{ before } |(\beta, 0) + z(t, \omega)| \geqq R) = \log(R/\beta)/\log(R/\rho). \tag{11}$$

Thus (11) gives an explicit expression for the probability that $L((\beta,0),\omega)$, which[2] passes with probability 1 through both circles

$$|(\beta,0) + z(t,\omega)| \leqslant \rho \text{ and } |(\beta,0) + z(t,\omega)| \leqslant R,$$

encounters the first circle before it does the second.

Proof. For a point ζ in the plane let $u(\zeta) = Pr(|\zeta + z(t,\omega)| \leqslant \rho \text{ before } |\zeta + z(t,\omega)| \geqq R)$, then[3] $u(\zeta)$ is a harmonic function of ζ in the ring $\rho \leqslant |\zeta| \leqslant R$ and its boundary values are $u(\zeta) = 0$ for $|\zeta| = R$ and $u(\zeta) = 1$ for $|\zeta| = \rho$. Consequently

$$u(\zeta) = \frac{\log(R/|\zeta|)}{\log(R/\rho)}, \quad \rho \leqslant |\zeta| \leqslant R.$$

LEMMA 3. *For every* $0 < \rho < 1$ *we have**

$$p(\rho) < c_1/\log(1/\rho). \tag{12}$$

Proof. By Lemmas 1 and 2 we have

$$p(\rho) \leqslant Pr(|(1,0) + z(t,\omega)| \leqslant \rho \text{ before } |(1,0) + z(t,\omega)| \geqq 2) +$$
$$+ Pr\left(\sup_{0\leqq t\leqq 1} |(1,0) + z(t,\omega)| \geqq 2\right) \cdot Pr\left(\inf_{0\leqq t\leqq 1} |(2,0) + z(t,\omega)| \leqslant \rho\right) =$$
$$= \log 2/\log(2/\rho) + \delta p(\rho, 2) \leqslant \log 2/[\log 2 + \log(1/\rho)] + \delta p(\rho)$$

with

$$\delta = Pr\left(\sup_{0\leqq t\leqq 1} |(1,0) + z(t,\omega)| \geqq 2\right) < 1.$$

Hence

$$p(\rho) \leqslant \frac{1}{1-\delta} \cdot \frac{\log 2}{\log 2 + \log(1/\rho)}, < \frac{1}{1-\delta} \cdot \frac{\log 2}{\log(1/\rho)}$$

i.e., (12).

* $c_1, c_2, \ldots, c_{17}$ are finite positive constants.

LEMMA 4. *For every* $0<\rho<\beta<1$ *we have*

$$p(\rho,\beta)<[c_1+\log(1/\beta)]/\log(1/\rho). \tag{13}$$

Proof. As in the proof of Lemma 3 we have

$$p(\rho,\beta)\leqslant Pr(|(\beta,0)+z(t,\omega)|\leqslant\rho \text{ before } |(\beta,0)+z(t,\omega)|\geqq 1)+Pr(\sup_{0\leqq t\leqq 1}|(\beta,0)+$$
$$+z(t,\omega)|\geqq 1)\cdot Pr(\inf_{0\leqq t\leqq 1}|(1,0)+z(t,\omega)|\leqslant\rho)<\log(1/\beta)/\log(1/\rho)+p(\rho),$$

and (13) follows from (12).

LEMMA 5. *For* $0<\rho<1/2$ *we have*

$$p(\rho)>c_2/\log(1/\rho). \tag{14}$$

Proof. Let $N>1$ be an integer and put

$$t_i=1/2+i/2N \quad (i=1,\ldots,N). \tag{15}$$

Let $E_i=\{\omega \mid |(1,0)+z(t_i,\omega)|\leqslant\rho\}$ be the event that the Brownian motion starting at distance 1 from the origin be within distance ρ from the origin at the time t_i. Let p_i denote the probability that E_i occur and p_{ij} denote the probability that both E_i and E_j occur $(i,j=1,\ldots,N)$. From (2), (4) and (15) we have

$$p_i=(1/2\pi t_i)\iint_{u^2+v^2\leqq\rho^2} e^{-((u-1)^2+v)^2/2t_i}\,du\,dv$$
$$>(1/2\pi)\cdot\pi\rho^2\cdot e^{-(\rho+1)^2-\rho^2}$$

or

$$p_i>c_3\rho^2. \tag{16}$$

Similarly, we have for $1\leqslant i<j\leqslant N$

$$p_{ij}\leqslant p_i\cdot Pr(|z(t_j,\omega)-z(t_i,\omega)|\leqslant 2\rho)=p_i\cdot\left(1/2\pi(t_j-t_i)\right)\iint_{u^2+v^2\leqq 4\rho^2} e^{-(u^2+v^2)/2(t_j-t_i)}\,du\,dv$$

and

$$p_{ij}<c_4[N/(j-i)]\rho^2 p_i. \tag{17}$$

Now, the events E_i all imply that $L_{0,1}((1,0);\omega)$ has points within distance ρ from the origin, an event whose probability was defined by (9) and (10) as $p(\rho)$. Hence, by (17),

$$p(\rho)\geqq\sum_{i=1}^{N}p_i-\sum_{1\leqq i<j\leqq N}p_{ij}$$
$$=\sum_{i=1}^{N}p_i\left(1-\sum_{j=i+1}^{N}p_{ij}\right)>\sum_{i=1}^{N}p_i\left(1-c_4\rho^2N\sum_{j=i+1}^{N}\frac{1}{j-i}\right)\geqq\left(1-c_4\rho^2N\sum_{j=1}^{N-1}\frac{1}{j}\right)\sum_{i=1}^{N}p_i$$

whence

$$p(\rho)>(1-c_5\rho^2N\log N)\sum_{i=1}^{N}p_i. \tag{18}$$

Let $c' = 1/(1+8c_5)$ and put

$$N = \left[c'/\rho^2 \log(1/\rho)\right] \tag{19}$$

(the square brackets denoting the integral part).

Then we have, for $0 < \rho < 1/e$,

$$\rho^2 N \log N \leqslant [c'/\log(1/\rho)] \cdot 2\log(1/\rho) < 1/2c_5. \tag{20}$$

Let now $\rho_0 > 0$ be such that $\rho_0 < 1/e$ and satisfy furthermore the condition $2\rho_0^2 < c'$. Then for every $0 < \rho \leqslant \rho_0$ the integer N calculated by (19) is greater than 2, we may thus apply (18) and (20) to obtain

$$p(\rho) > 1/2 \sum_{i=1}^{N} p_i \qquad (0 < \rho < \rho_0).$$

Thus, by (16) and (19),

$$p(\rho) > (c_3/2)\,\rho^2\left[c'/\rho^2 \log(1/\rho)\right] > c_6/\log(1/\rho)$$

for $0 < \rho \leqslant \rho_0$. Since $p(\rho) > p(\rho_0) > 0$ for $\rho > \rho_0$, (14) is valid for $0 < \rho < 1/2$ and the Lemma is established.

LEMMA 6. *For every* $0 < \rho < 1$ *we have*

$$Pr\left(\inf_{1/2 \leqq t \leqq 1} |z(t,\omega)| \leqslant \rho\right) < c_7/\log(1/\rho). \tag{21}$$

Proof. From (2), (4), (13) and the isotropy of the Brownian movement we have

$$Pr\left(\inf_{1/2 \leqq t \leqq 1} |z(t,\omega)| \leqslant \rho\right) \leqslant Pr(|z(1/2,\omega)| \leqslant \rho) + \int_{\rho}^{\infty} p(\rho,\beta)\, d_{\beta} Pr(|z(1/2,\omega)| \leqslant \beta)$$

$$< (1/\pi) \iint_{u^2+v^2 \leqq 1/4} e^{-u^2-v^2}\, du\, dv + \int_{\rho}^{\infty} \frac{c_1 + \log(1/\beta)}{\log(1/\rho)} \cdot 2\beta e^{-\beta^2}\, d\beta$$

$$< \rho^2 + [2/\log(1/\rho)] \int_{0}^{\infty} [c_1 + \log(1/\beta)]\beta e^{-\beta^2}\, d\beta$$

$$= \rho^2 + c_8/\log(1/\rho)$$

since the last integral is convergent. This proves (21) for $0 < \rho < 1$.

LEMMA 7. *For every* $0 < \rho < 1/2$ *we have*

$$Pr\left(\inf_{1/2 \leqq t \leqq 1} |z(t,\omega)| \leqslant \rho\right) > c_9/\log(1/\rho). \tag{22}$$

Proof. By Lemma 1 and the homogeneity property we have

$$Pr\left(\inf_{1/2 \leqq t \leqq 1} |z(t,\omega)| \leqslant \rho\right) \geqq Pr(|z(1/2,\omega)| \leqslant 1/\sqrt{2}) \cdot Pr\left(\inf_{0 \leqq t \leqq 1/2} |(1/\sqrt{2},0) + z(t,\omega)| \leqslant \rho\right) = c_{10} p(\sqrt{2}\,\rho).$$

In view of (14), (22) follows for $0 < \rho < 1/(2\sqrt{2})$ and hence also for $0 < \rho < 1/2$.

The next Lemma is rather complicated, but it is quite close to the theorem we wish to prove.

LEMMA 8. *Let* k>1 *be a fixed positive integer. Let* H *be a positive number and* n *a positive integer and let*

$$\rho = e^{-Hn^{2/k}} \tag{23}$$

For $\nu_1, \nu_2 = 1, 2, \ldots, n$ *put* $\nu = (\nu_1 - 1)n + \nu_2$, *and, for* $\nu = 1, \ldots, n^2$, *let*

$$z_\nu = \left((1/5) + (\nu_1/2n), (1/5) + (\nu_2/2n)\right) \tag{24}$$

and let $S_\nu(\rho)$ *denote the circle* $\{z \mid |z - z_\nu| < \rho\}$. *Let* $F_\nu = F_\nu(H)$ *be the event that there exist* k *numbers* t_i, $i = 1, \ldots, k$ *satisfying*

$$0 < t_1 \leqslant 1, \quad 1/2 \leqslant t_i - t_{i-1} \leqslant 1 \quad (i = 2, \ldots, k) \tag{25}$$

for which

$$x(t_i, \omega) \in S_\nu(\rho) \quad (i = 1, \ldots, k).$$

Let $F^{(n)} = F^{(n)}(H)$ *be the union of the events* F_ν, $\nu = 1, \ldots, n^2$.

Then

$$\liminf_{n=\infty} Pr(F^{(n)}) > c_{11}/H^k \tag{26}$$

for all sufficiently large H.

Proof. Let q_ν be the probability that F_ν occur and $q_{\nu,\nu'}$ be the probability that both F_ν and $F_{\nu'}$ occur ($\nu, \nu' = 1, 2, \ldots, n^2$).

We may assume $H > 1$ so that $\rho < 1/e$ and the estimates of the previous Lemmas become applicable. Since $|z_\nu| < 1$ we have from Lemmas 1, 5, and 7*

$$q_\nu \geqq Pr\left(\inf_{0 < t \leqslant 1} |z(t,\omega) - z_\nu| < \rho\right) \cdot [1/3 \cdot Pr(\inf_{1/2 \leqq t \leqq 1} |z(t,\omega)| < \rho)]^{k-1}$$

$$> [c_6/\log(1/\rho)] \cdot [1/3 \cdot (c_9/\log(1/\rho)]^{k-1}.$$

Since, by (23), $[\log(1/\rho)]^k = H^k n^2$, this gives

$$q_\nu > c_{12}/H^k n^2 \ (\nu = 1, 2, \ldots, n^2). \tag{27}$$

For $\nu \neq \nu'$ let P_1 be the probability that $L_{0,1}(\omega)$ pass through at least one of the circles $S_\nu(\rho)$ and $S_{\nu'}(\rho)$; let P_2 be an upper bound for the probability that $L_{0,1}(z;\omega)$ with $z \in S_\nu(\rho)$ encounter $S_{\nu'}(\rho)$; and let P_3 be an upper bound for the probability that a Brownian motion starting at $z' \in S_\nu(\rho)$ encounters $S_{\nu'}(\rho)$ again for some $1/2 \leqslant t \leqslant 1$. By obvious symmetry considerations we have for $\nu \neq \nu'$ the inequality

$$q_{\nu,\nu'} < P_1(P_2 + P_3)^{2k-1}.$$

Now by Lemmas 1 and 4 we have

$$P_1 < p(\rho, |z_\nu|) + p(\rho, |z_{\nu'}|) \leqslant 2p(\rho, |z_1|) < 2[c_1 + \log(1/|z_1|)]/\log(1/\rho).$$

* The conditional probability of being in $S_\nu(\rho)$ at time t_i, given that the path is at $z' \in S_\nu(\rho)$ at time t_{i-1}, is $\geqq$ the probability that $z' + x(t_i - t_{i-1}, \omega)$ is in the intersection of $S_\nu(\rho)$ and a circle of radius ρ about z'; this intersection contains a sector of opening $2\pi/3$ and hence, because of the isotropy property, is greater than the second factor.

Also

$$P_2 < p(\rho, |z_\nu - z_{\nu'}| - 2\rho).$$

Now, by (24), $|z_\nu - z_{\nu'}| \geq 1/2n$ while, by (23), we have $\rho < 1/4n$ for all $n > n_0 = n_0(k)$. Hence for all $n > n_0$ we have from (13)

$$P_2 < [c_1 + \log(2/|z_\nu - z_{\nu'}|)]/\log(1/\rho).$$

Finally, by (21),

$$P_3 < Pr\left(\inf_{1/2 \leq t \leq 1} |z(t,\omega)| \leqslant 2\rho\right) < c_7/\log(1/2\rho).$$

Combining these estimates, we have for $n > n_0$ and $\nu \neq \nu'$

$$q_{\nu,\nu'} < c_{13}\left\{[1 + \log(1/|z_\nu - z_{\nu'}|)]/\log(1/\rho)\right\}^{2k},$$

or, by (23) and (24),

$$q_{\nu,\nu'} < [c_{14}/H^{2k} n^4] \log^{2k}(1/|z_\nu - z_{\nu'}|). \tag{28}$$

From (27) and (28) we have

$$Pr(F^{(n)}) \geq \sum_{\nu=1}^{n^2} q_\nu - \sum_{1 \leq \nu < \nu' \leq n^2} q_{\nu,\nu'}$$

$$> c_{12}/H^k - c_{14}/H^{2k} \cdot 1/n^4 \cdot \sum_{1 \leq \nu < \nu' \leq n^2} \log^{2k}(1/|z_\nu - z_{\nu'}|).$$

Now this last sum is smaller than twice

$$n^2 \sum_{\nu=2}^{n^2} \log^{2k}(1/|z_\nu - z_1|) < n^2 \sum_{\nu=2}^{n} (2\nu+1) \log^{2k}(1/|z_\nu - z_1|)$$

$$= n^2 \sum_{\nu=2}^{n} (2\nu+1) \log^{2k}[2n/(\nu-1)]$$

$$< 5n^2 \sum_{j=1}^{n} j \log^{2k}(2n/j)$$

$$< 5n^3 \max_{1 \leq u \leq n} u \log^{2k}(2n/u)$$

$$< c_{15} n^4$$

(c_{15} depends on k, but k is fixed throughout).

Hence

$$Pr(F^{(n)}) > c_{12}/H^k - c_{16}/H^{2k} \tag{29}$$

for all $n > n_0$. Taking $H^k > 2c_{16}/c_{12}$ we obtain (26).

Proof of Theorem 1. Let $k > 1$ be fixed. For every $\omega \in \Omega$ let

$$g_k(\omega) = \inf \sum_{i=2}^{k} |z(t_i, \omega) - z(t_{i-1}, \omega)|$$

where the *inf* is taken over all sequences $t_1, \ldots, t_k$ satisfying (25). Since the path is continuous for almost all ω, $g_k(\omega)$ is easily seen to be a random variable. As the event $F^{(n)}$ of Lemma 8 implies $g_k(\omega) < 2k\,\rho$ and since $\rho \to 0$ as $n \to \infty$ we have

$$Pr(g_k(\omega) < \varepsilon) > c_{17} > 0$$

for every $\varepsilon > 0$ (where c_{17} is, of course, independent of ε). Hence, $Pr(g_k(\omega) = 0) \geqq c_{17}$ which implies, for all continuous paths, the existence of a k-multiple point of $L_{0,n}(\omega)$. For every integer $j = 1, 2, \ldots$ let G_j denote the event $L_{(j-1)k,jk}(\omega)$ has a k-multiple point. Then $Pr(G_j) = Pr(G_1) > 0$ and the events G_j are independent. Therefore, with probability 1 infinitely many of the events G_j occur, and hence there is probability 1 that $L(\omega)$ have k-multiple points. q. e. d.

Using the homogeneity property of the Brownian motion, we deduce immediately

THEOREM 2. *Let* a, b *be any positive numbers with* $0 \leqslant a < b < \infty$ *then, with probability* 1, *the* (a, b)*-path* $L_{a,b}(\omega)$ *has multiple points of arbitrarily high* (*finite*) *multiplicity.*

It also follows that there exist k-multiple points for which the intervals between returns to the point are arbitrarily large.

In view of the fact[4] that, for almost all ω, $L(\omega)$ is dense in the entire plane we have from Theorem 2 the following

THEOREM 3. *For almost all* ω *the set of* k*-multiple points of* $L(\omega)$ *is dense everywhere in the plane for all* $k = 2, 3, \ldots$.

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SECOND ORDER EFFECTS IN ELASTICITY AND HYDRODYNAMICS

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0. The classical mechanics of continua is built upon two ideal materials of extreme behaviour, namely (i) the *Hookean solid*, the rheological equation of which is

$$p_{ij} = \lambda \varepsilon_{\alpha\alpha} \delta_{ij} + 2\mu\varepsilon_{ij} \qquad (0.1)$$

and in which all stress work is completely conserved. (ii) the *Newtonian liquid*, with the rheological equation

$$p_{ij} = -p\,\delta_{ij} + 2\eta f_{ij} \qquad (0.2)$$

in which all stress work is entirely dissipated. In these equations

$$\varepsilon_{ij} = (\partial u_i/\partial x_j + \partial u_j/\partial x_i)/2 \qquad (0.3)$$

is the *strain-tensor*, and

$$f_{ij} = (\partial v_i/\partial x_j + \partial v_j/\partial x_i)/2 \qquad (0.4)$$

the *flow-tensor* (also named rate of strain-tensor).

For (0.3) it is stipulated that the displacement gradients are infitesimal, for (0.4) this is implied by the definition of velocity. The strain ε is therefore "infinitesimal", while the flow f will generally be finite.

Applying to the stress tensor, as expressed by either (0.1) or (0.2), the momentum equations, all "first order" phenomena are solvable in principle.

1. Second order phenomena arise in classical *elasticity* as soon as one proceeds to the consideration of finite displacement gradients. Instead of (0.1) one can write

$$p_{ij} = F_0 \delta_{ij} + F_1 e_{ij} \qquad (1.1)$$

but the finite strain e_{ij} cannot be defined by an expression analogous to (0.3). The usual definition of finite strain is based upon the expression for the strained squared linear element ds^2 which is put in relation to the unstrained element ds_0^2. Two measures of strain result, one "Lagrangean" derived from $(ds^2-ds_0^2)/ds_0^2$ first postulated by Green*; the other "Eulerian" from $(ds^2-ds_0)/ds^2$, first postulated by Almansi*. However, in a more general manner we may consider the strain ellipsoid with half axes $\lambda(i)$. Any function of $\lambda(i)$ which vanishes for $\lambda(i) = 1$ and is reduced to (0.3) for small $\lambda(i)-1$, will form a suitable expression of a Lagrangean strain. By considering the reciprocal strain-ellipsoid with half axes $l(i) = 1/\lambda(i)$ we get expressions for "Eulerian" strains. We thus find, for instance, the four measures

* For references compare the exhaustive survey by Truesdell[1].

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$$\left.\begin{aligned} e^{C}(i) &= \lambda(i) - 1 \\ e^{G}(i) &= \tfrac{1}{2}[\lambda^{2}(i) - 1] \\ e^{A}(i) &= \tfrac{1}{2}[1 - l^{2}(i)] \\ e^{S}(i) &= 1 - l(i) \end{aligned}\right\} \qquad (1.2)$$

the first postulated by Cauchy for infinitesimal strain but here generalized for finite strain, the last postulated by Swainger [20]. These measures are asymmetrical.

A symmetrical measure was proposed by Hencky[2] with

$$e^{H}(i) = ln\,\lambda(i) = -ln\,l(i) \qquad (1.3)$$

In order to appreciate the differences between these measures, let the length of a rod be doubled; then the different measures will give

$$e^{G} = 100\%,\ e^{C} = 150\%,\ e^{H} = 66\%,\ e^{A} = 37.5\%,\ e^{S} = 50\%$$

while when its length is halved they are

$$e^{C} = -50\%,\ e^{G} = -37.5\%,\ e^{H} = -66\%,\ e^{A} = -150\%,\ e^{S} = -100\%.$$

It is clear that if the elastic response of some material can adequately be described by a linear relation such as (1.1) when some such measure e is used, the application of any other measure will result in a non-linear relation with second and higher order terms. It should also be noted that the "coefficients" F_0 and F_1 will generally be functions of the invariants of the tensor e_{ij}. This may also introduce non-linear relations in the expressions relating *components* of the stress- and strain-tensor. These two kinds of second order effects are *trivial*. The subject of the present paper is formed by entirely new second order phenomena which arise when the stress- and strain-tensors in their mutual relations are considered.

2. This can best be illustrated on the example of simple shear dealt with by Love[3] in Art 37. Simple shear is given kinematically by the equations (in Love's notation)

$$x_1 = x + sy, \qquad y_1 = y, \qquad z_1 = z \qquad (2.1)$$

Introducing for the displacement-gradient s

$$s = 2\tan\alpha \qquad (2.2)$$

Love calculates

$$\lambda_1 = (1 - \sin\alpha)/\cos\alpha, \quad \lambda_2 = (1 + \sin\alpha)/\cos\alpha, \quad \lambda_3 = 1 \qquad (2.3),$$

and he proves that the directions of the principal axes of strain are the bisectors of the angle $\pi/2 + \alpha$ with the x-axis, and the angle through which the principal axes are turned is the angle α. The stress caused by the strain will have the principal components from (1.1)

$$p(i) = F_0 + F_1 e(i) \qquad (2.4).*$$

The components of stress with respect to the system x, y, z, will be from Love's equations, Art. 49:

* We write the index i in brackets to indicate that $p(i)$ and $e(i)$ are not tensors.

$$\left.\begin{aligned} p_{xx} &= {}^1/_2 \cdot (p_1 + p_2) - {}^1/_2 \cdot (p_1 - p_2) \sin \alpha \\ p_{yy} &= {}^1/_2 \cdot (p_1 + p_2) + {}^1/_2 \cdot (p_1 - p_2) \sin \alpha \\ p_{zz} &= p_3 \\ p_{xy} &= {}^1/_2 \cdot (p_1 - p_2) \cos \alpha, \quad p_{yz} = p_{zx} = 0 \end{aligned}\right\} \qquad (2.5).$$

Introducing the expressions for the principal stresses from (2.4) into (2.5) gives

$$\left.\begin{aligned} p_{xx} &= F_0 + {}^1/_2 \cdot F_1 [(e_1 + e_2) - (e_1 - e_2) \sin \alpha] \\ p_{yy} &= F_0 + {}^1/_2 \cdot F_1 [(e_1 + e_2) + (e_1 - e_2) \sin \alpha] \\ p_{xy} &= -{}^1/_2 \cdot F_1 (e_1 - e_2) \cos \alpha, \quad p_{yz} = p_{zx} = 0 \end{aligned}\right\} \qquad (2.6)$$

Now from (2.3,3), $e_3=0$ and therefore $F_0=p_{zz}$. We then introduce the five different measures of (1.2) and (1.3) and find the stress components as entered in the following table (expressing the goniometrical functions in terms of s of (2.2)).

	C $e = \lambda(i)-1$	G $e = {}^1/_2 \cdot [\lambda^2(i)-1]$	H $e = ln\ \lambda(i)$	A $e = {}^1/_2 \cdot [1-I^2(i)]$	S $e = 1-I(i)$
$(p_{xx}-p_{zz})/F_1$	$(1+s^2/2)/\alpha-1$	$s^2/2$	$\frac{s}{4\alpha}\, ln \frac{\alpha+s/2}{\alpha-s/2}$	0	$1-1/\alpha$
$(p_{yy}-p_{zz})/F_1$	$1/\alpha-1$	0	$-\frac{s}{4\alpha}\, ln \frac{\alpha+s/2}{\alpha-s/2}$	$-s^2/2$	$1-(1+s^2/2)/\alpha$
p_{xy}/F_1	$s/2\alpha$	$s/2$	$\frac{1}{2\alpha}\, ln \frac{\alpha+s/2}{\alpha-s/2}$	$s/2$	$s/2\alpha$
$(p_{\alpha\alpha}/3-p_{zz})/F_1$	$^2/_3 \cdot (\alpha-1)$	$s^2/6$	0	$-s^2/6$	$^2/_3 \cdot (1-\alpha)$

$\alpha = (1-s^2/4)^{1/2}$

The factor F_1 can be identified with the shear-modulus μ, but it is connected with entirely new second order effects of two kinds:

(i) in order to produce simple shear, an isotropic tension is required in the case of the Lagrangean measures and an isotropic compression in the case of the Eulerian measures. This may be called the *Kelvin effect.*

(ii) There is a tensile stress in the direction of the displacement (x) and a compressive stress in the direction normal to it (y). This may be called the *Poynting effect.*

The first of these designations is based upon a remark by William Thomson (Lord Kelvin)[4] as follows: "It is possible that a shearing stress may produce in a truly isotropic solid condensation or dilatation in proportion to the square of its value".

3. For the explanation of the second designation we consider the torsion of a cylindrical rod. Let us use both the Green and Almansi measures. Let l be the length of the cylinder and Ω its twist, so that Ω/l is the twist per unit length. Let us make experimental arrangements such as were carried out by Rivlin[5] so that the length is kept constant.

This is thus a case analogous to the simple shear considered in the preceding section. We accordingly find the strain-tensors

$$\overset{G}{e}_{ij} = r\Omega/2l \cdot \begin{Vmatrix} 0 & 0 & 0 \\ 0 & r\Omega/l & 1 \\ 0 & 1 & 0 \end{Vmatrix} \tag{3.1}$$

$$\overset{A}{e}_{ij} = r\Omega/2l \cdot \begin{Vmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -r\Omega/l \end{Vmatrix} \tag{3.2}$$

and the stress tensors

$$\overset{G}{p}_{ij} - \overset{G}{F}_0 \delta_{ij} = \overset{G}{F}_1 r\Omega/2l \cdot \begin{Vmatrix} 0 & 0 & 0 \\ 0 & r\Omega/l & 1 \\ 0 & 1 & 0 \end{Vmatrix} \tag{3.3}$$

$$\overset{A}{p}_{ij} - \overset{A}{F}_0 \delta_{ij} = \overset{A}{F}_1 r\Omega/2l \cdot \begin{Vmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -r\Omega/l \end{Vmatrix} \tag{3.4}.$$

We now apply the first momentum equation of equilibrium (the only one which does not vanish identically)

$$dp\,(rr)/dr + [p\,(rr) - p(\theta\theta)]/r = 0 \tag{3.5},$$

and find by integration, assuming that the sides $r = R$ are stress-free,

$$\overset{G}{p}(zz) = -\overset{G}{F}_1 \Omega^2/4\, l^2 \cdot (R^2 - r^2) \tag{3.6}$$

$$\overset{A}{p}(zz) = -\overset{A}{F}_1 r^2 \Omega^2/2\, l^2 \tag{3.7}$$

In both cases we find that a pressure acting upon the bases of the cylinder is required in order to keep its length constant. Conversely, it may be concluded that in the absence of such pressures the cylinder would be extended in simple torsion. Such an effect was observed by Poynting[6,7] in steel wires. Incidentally, Poynting observed at the same time also a Kelvin effect.

Similar results can be derived by using any other measure of strain and it is remarkable that these two second order effects are also present when the measure is linear, such as (1.2,1) and (1.2,4).

4. Second order effects of this kind cannot arise in classical *viscosity*. In viscous flow the amount of finite deformation reached at any time is of no physical significance. As a matter of fact no definite meaning can be attached to such deformation because, in contradistinction to elasticity, while there exists an "unstrained state", no "undeformed state" can be defined. The viscous resistance at any moment t depends upon the gradient of the infinitesimal displacement during the time-element dt following t.

Here second-order effects result from a generalization of (0.2).

Starting from a general tensor relation

$$p_{ij} = F_0 \delta_{ij} + F_1 f_{ij} + F_2 f_{i\alpha} f_{\alpha j} + F_3 f_{i\alpha} f_{\alpha\beta} f_{\beta j} + \ldots \tag{4.1}$$

Reiner[8] has shown* that by applying the Cayley-Hamilton relation, this series is reduced to three terms, namely

$$p_{ij} = F_0 \delta_{ij} + F_1 f_{ij} + F_2 f_{i\alpha} f_{\alpha j} \tag{4.2}$$

Truesdell[1] has named a fluid following eq. (4.2) "Reiner-fluid". Rivlin[9] has specialized (4.2) for the case of incompressibility and Truesdell uses for such ideal material the designation Rivlin-fluid. Second order effects arise from the third term. Considering laminar flow between two parallel platens at distance D with velocity gradient $\dot{s}$

$$\dot{s} = dv/dy = V/D \tag{4.3}$$

the flow tensor is, in accordance with (0.4)

$$f_{ij} = \dot{s}/2 \cdot \begin{Vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{Vmatrix} \tag{4.4}$$

and therefore from (4.2)

$$p_{ij} = F_0 \delta_{ij} + F_1 \dot{s}/2 \cdot \begin{Vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{Vmatrix} + F_2 \dot{s}^2/4 \cdot \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{Vmatrix} \tag{4.5}$$

Note that the classical methods of viscometry in which shearing stress and rate of shear are related cannot reveal the existence of "cross stresses". Assuming the material open to air pressure, taken as zero, at the sides of the "strip", F_0 vanishes and we have

$$\left.\begin{aligned} p_{xx} &= p_{yy} = F_2 \dot{s}^2/4 \\ p_{xy} &= p_{yx} = F_1 \dot{s}/2 \\ p_{zz} &= p_{yz} = p_{zx} = 0 \\ p_{\alpha\alpha}/3 &= F_2 \dot{s}^2/9 \end{aligned}\right\} \tag{4.6}$$

The factor F_1 can be identified with the shear viscosity η of (0.2), but there appears in the factor F_2 a new material parameter which has been named *coefficient of cross-viscosity*. Here also two second order effects are present. They are:

(i) in order to maintain simple shear, an isotropic tension or compression is necessary, in accordance with the sign of F_2,

(ii) there are "cross-stresses" in the direction of the flow (x) and in the direction normal to it (y), both having *the same sign*.

These second order effects are very different from those encountered in finite "classical" elasticity. While there they are related to a "classical" material parameter, the shear modulus μ, here they result from a new material parameter which is unknown to classical hydrodynamics. Reynolds found a volume change through shear in such a "fluid" as wet sand, and we may denote the first effect by "dilatancy", a term coined by him.

Regarding the second effect, we consider the analogy to the torsion of an elastic cylinder, namely the case of torsional flow

$$v_\theta = r\omega \tag{4.7}$$

* Following a suggestion by Racah.

between two platens at distance H, one at rest while the other rotates with the angular velocity.

One easily finds (compare Braun and Reiner[22]) from the second momentum equation

$$\omega = \Omega\,(1 - z/H) \tag{4.8}$$

and

$$p_{ij} = F_0\,\delta_{ij} - r\,\Omega\eta/H \cdot \begin{Vmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{Vmatrix} + \pi^2\,\Omega^2 F_2/4H^2 \cdot \begin{Vmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Vmatrix} \tag{4.9}$$

Assuming that at the edge $r = R$ the liquid has a free surface, the first momentum equation yields

$$p\,(rr) = F_0 = -\,(\Omega^2/8H^2)\,(R^2 - r^2)\,F_2 \tag{4.10}$$

from which

$$p\,(zz) = -\,(\Omega^2/8H^2)\,(R^2 - 3r^2)\,F_2 \tag{4.11}.$$

The second order "cross-effects" accordingly consist of radial pressures $p\,(rr)$ and pressures normal to the platens $p\,(zz)$. Therefore, if the upper platen has an opening in the centre to which a vertical tube is attached, the liquid will be forced by the pressure $p\,(rr)$ to flow radially towards the centre against the action of centrifugal forces; and arriving at the centre, it will be forced by the pressure $p\,(zz)$ to rise in the tube against the action of gravity. The whole arrangement thus forms a sort of centripetal pump.

5. The position with regard to elasticity as described in Sections 1 to 3 above cannot be considered as satisfactory. As was shown, the assumption of one or other of possible measures of strain carries with it a definite distribution of stresses which may or may not be confirmed by experiment.

For instance, Rivlin[5], in the experiment mentioned above, found that the normal pressure $p\,(zz)$ over the plane ends of the cylinder was largest at the centre, falling off parabolically towards the edge where it has a finite value. As follows from (3.6) and (3.7), this is not what results when either the Green or the Almansi-measure is used. Rivlin succeeded in describing the results in terms of another measure proposed by Mooney. This situation points, however, to the fact that the rheological equation (1.1) is not general enough. It has been generalized by Reiner (1948) on the model of (4.2) to

$$p_{ij} = F_0\,\delta_{ij} + F_1 e_{ij} + F_2 e_{i\alpha}\,e_{\alpha j} \tag{5.1}$$

This rheological equation permits of an adequate description of any experimentally found stress-distribution, using any measure of strain. For instance, using the Almansi-measure, we have for the case described in Section 3 above

$$p_{ij} = F_0\,\delta_{ij} + F_1(r/2l) \begin{Vmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -(r\,\Omega/l) \end{Vmatrix} + F_2(r^2\,\Omega^2/4l^2) \begin{Vmatrix} 0 & 0 & 0 \\ 0 & 1 & -(r\,\Omega/l) \\ 0 & -(r\,\Omega/l) & 1 + (r^2\,\Omega^2/l^2) \end{Vmatrix} \tag{5.2}$$

and integrating the momentum-equations under the assumption, as before, that the sides are stress-free

$$p\,(zz) = -\left\{F_1\,(\Omega^2/4l^2)\,(R^2-r^2) + F_2(\Omega^2/8l^2)\,[R^2-3r^2-(\Omega/2l^2)\,(R^4-r^4)]\right\} \quad (5.3)$$

$$\text{Now for } r = 0, \quad p\,(zz)^{(0)} = -(\Omega^2R^2/4l^2)\,\left\{F_1 + F_2\,[1-(\Omega R^2/2l^2)]/2\right\} \quad (5.4)$$

$$\text{while for } r = R, \quad p\,(zz)^{(R)} = F_2\,(\Omega^2R^2/4l^2) \quad (5.5)$$

These two values are sufficient for determining both F_1 and F_2. F_2 turns out to be negative.

One could assume any other measure of strain with similar treatment and the generalized equation of elasticity (5.1) therefore does not prejudice experimental results.

6. The phenomena described above have recently come to the fore in rheology in connection with investigations on the rheological behaviour of various two-phase systems which cannot be considered simply as either elastic solids or viscous liquids, but partake in both properties of elasticity and viscosity. True, Poynting's observations on steel within the elastic limit mentioned above refer to elasticity as such, but they fell entirely into oblivion and they have not been repeated since then. Swift[10] has observed lengthening of metal rods in plastic torsion and I suggest that this "cross-plasticity" is due to the successive "freezing in" of cross-elastic strains. Both Poynting's and Swift's observations wait for a theoretical analysis in terms of second-order cross-behaviour.* At the other extreme, no case of cross-viscosity of a simple liquid is known. Truesdell[12] has argued from dimensional considerations that cross-viscosity should be present in rarefied air and Reiner[13] has analysed a "centripetal pump effect" in air in such terms, but his theory requires experimental verification. However, cross-phenomena were most strikingly first demonstrated by Weissenberg at the 1946 meeting of the British Rheologists' Club (compare Weissenberg[14]), among them the centripetal pump effect mentioned in Section 4 above, on a variety of *colloidal solutions*. They were confirmed by Reiner, Scott Blair and Hawley[15] for gelatinized condensed milk and by Lax-Weiner and Schoenfeld-Reiner[16] for blown bitumen. For such materials the present theory embodied in the two equations (4.2) and (5.1) must be carried further to one in which both are combined. This can be done in two different ways, considering whether the materials are either (i) sols or (ii) gels (compare Reiner[17]).

(i) for sols the rate of deformation d is additive, or

$$d_{ij} = f_{ij} + \dot{e}_{ij} \quad (6.1)$$

where f_{ij} and $\dot{e}_{ij}$ must be expressed from reversions of (4.2) and (5.1) as functions of the stress p_{ij} of the system.

(ii) For gels the stress p_{ij} of the system is the sum of two partial stresses, each one expressed by (4.2) and (5.1) respectively, in which f_{ij} is identified with $\dot{e}_{ij}$. These procedures have been indicated by Oldroyd[18].

Observations by Braun, Frei and Reiner[19] suggest that some materials, when fresh, may be of the gel type with the elastic component more pronounced and that they are converted through continuous shear into sols with the viscous component preponderant.

However, very few quantitative observations have been published so far in this field. Greensmith and Rivlin[23] have claimed to have proven experimentally the

* Prager[11] has used an equation similar to (5.1) for plastic deformations.

existence of a cross-viscosity effect in solutions of polyisobutylene in *o*-chlorobenzene, but Roberts[21] claims to have repeated their experiments with contradicting results.

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SOME THEOREMS CONCERNING ASSOCIATIVE ZORN RINGS

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I. PRELIMINARIES

Whereas algebras of finite dimensionality over a field Φ have been the object of extensive investigations for more than fifty years, one finds in the literature only scattered results concerning infinite algebras (i.e. algebras with infinite dimensionality). Of particular interest among the latter are the so called *algebraic algebras*, i.e. algebras whose elements satisfy algebraic equations of the form $f(x) = 0$, where $f(x)$ is a polynomial of some positive degree, with coefficients in the reference field Φ. For some recent results concerning this class of algebras, consult papers [1, 2, 4, 5, 6, 9, 12, 13, 14]. Actually, in some of these papers algebraic algebras are dealt with in the much more general frame of the class of the so-called *I-rings*, i.e. associative rings in which every non-nil right ideal contains a nonzero idempotent*. An important subclass of the class of I-rings is the class of *faithful I-rings* (in short *FI-rings*), i.e. rings whose homomorphic images are I-rings. Thus every algebraic algebra is an FI-ring. An I-ring without nilpotent elements is called a *plain ring*.

The object of the present note is to show that some theorems on finite algebras carry over to rather extended classes of infinite algebraic algebras and — more generally — to I-rings. We do this by leaning heavily on some theorems which were derived by the author in a recent paper[13]. In the present section we shall restate these theorems for the convenience of the reader.

THEOREM 1. *If* S *is a semi-simple*** *I-ring and* $a \in S$ *a nilpotent element of index* n, *then the two-sided ideal* (a) *generated by* a *contains a complete system of* n^2 *matrix units.****

THEOREM 2. *If* T *is a plain ring with an identity element* e, *and* S *is the total* n *by* n *matrix ring over* T, *then any one sided inverse in* S *is two sided, i.e. from* ab = e *it follows that also* ba = e.

THEOREM 3. *Suppose that a ring* S *has two isomorphic representations as a total matrix ring over plain rings, with degrees* n *and* m. *Then* n = m.

Remark. Theorems 1, 2, 3 correspond to Theorem 2.1, Lemma 3.2 and Theorem 3.2 in ref. 13, respectively.

* This terminology was introduced by the author in his paper[13] and is retained in the present note for the sake of convenience. Previously, Kaplansky referred to these rings as *associative Zorn rings* — a fact which came to the author's notice when his paper[13] was already in print. It should be also mentioned that this class of rings has been singled out by G. Koethe in a paper which appeared in 1930 (compare ref. 8, p. 8).

** The notions semi-simplicity, primitivity, etc. are used here in the sense of Jacobson[3].

*** See definition at the end of the section.

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A ring S is said to be of *finite index* $i(S)$ if S contains nilpotent elements of index $i(S)$ but of no higher index. If the ring S is itself nilpotent and $S^n=0$, $S^{n-1} \neq 0$, we say that n is the *index of nilpotence* of S.

A system of m^2 elements e_{ik}, $i,k = 1; \ldots, m$ satisfying the relations

$$e_{ik}e_{k'j} \begin{cases} = 0 \text{ for } k \neq k' \\ = e_{ij} \text{ for } k = k' \end{cases} \tag{1}$$

shall be referred to as a *complete matrix units system.*

II. DEGREE AND INDEX

Total matrix rings of finite degree over a plain ring with an identity play an important role in the study of I-rings with finite index and of I-rings whose primitive[3] images are of finite index (compare [5, 6, 13, 14]). It is therefore desirable to make a closer study of these matrix rings. Let T be a plain ring with unit and $S=T_n$ the total matrix ring of degree n over T. In the special case where T is a division ring it is known that the index $i(T_n)$ of T_n is equal to the degree n of T_n. In the present section we show that this holds also in case T is an arbitrary plain ring with unit*. We need the following lemmas:

Lemma 1. *Let* $d_{ik}, i,k = 1, \ldots, m$ *be a complete set of matrix units in* $S = T_n$. *Then* $m \leqslant n$.

Proof. Suppose that $m > n$. Write $S = \sum_{i,k=1}^{n} e_{ik}$, where the e_{ik} constitute a basic complete set of matrix units of S over T. In particular, every e_{ik} is commutative with every $t \in T$. It follows that $0 \subset e_{11}Sd_{11}Se_{11} \subset e_{11}Se_{11} = e_{11}Te_{11}$. There is an element a_{11} such that $a_{11} \in e_{11}Sd_{11}$, $0 \subset a_{11}d_{11}Se_{11} \subseteq e_{11}Te_{11}$. Since $e_{11}Te_{11} \cong T$ and T is plain, so is $e_{11}Te_{11}$ and thus the nonzero right ideal $a_{11}d_{11}Se_{11}$ of $e_{11}Te_{11}$ contains a nonzero idempotent e_1 such that $e_1 = a_{11}b_{11}$, where $b_{11} \in d_{11}Se_{11}$. We put

$$a = \sum_{i=1}^{n} e_{i1}e_{11}a_{11}d_{1i}, \quad b = \sum_{i=1}^{n} d_{i1}b_{11}e_{1}e_{1i} \text{ and } e = ab \tag{2}$$

One verifies easily that e is an idempotent and since** e_i is in the centre of $e_{11}Te_{11}$ it follows that e is in the centre of S. As is easily seen, the two-sided ideal eS of S is a total matrix ring over the plain ring eT with the unit e. Hence from $e=ab$ it follows in view of Theorem 2 that also $ba = e$. By (1) we have $d_{n+11}e = d_{n+11}b_{11}e_1e_{11} \neq 0$, whereas $ed_{n+11} = 0$, which is impossible since e is in the centre of S. This contradiction shows that we must have $n \geqq m$, q. e. d.

Lemma 2. *If a ring* S *contains a complete system of* n^2 *matrix units* e_{ik}, $i,k = 1,\ldots, n$, *then* $i(S) \leqslant n$.

Proof. In fact, by (1) it follows easily that the element $\sum_{i=1}^{n-1} e_{ii+1}$ is nilpotent of index n.

THEOREM 4. *Let* $S = T_n$ *be a total matrix ring of degree* n *over a plain ring* T *possessing an identity element, and* $i(S)$ *the index of* S. *Then* $i(S) = n$.

Proof. By assumption we may put $S = \sum_{i,k=1}^{n} e_{ik}T$ where the e_{ik} constitute a basic complete

* The condition that T should have an identity element is dispensable.

** An idempotent in any ring without nilpotent elements other than zero belongs to the centre of the ring

matrix units system. By Lemma 2 we have $i(S) \geqq n$. The theorem will be proved if we show that $i(S) \leqslant n$. To this end let a denote an arbitrary nilpotent element in S. By Theorem 1 the ideal (a) generated by a in S contains a complete system of m^2 matrix units. Hence by Lemma 1 we have $m \leqslant n$, i.e. $i(S) \leqslant n$, which completes the proof of the theorem.

III. NILLITY AND NILPOTENCE

Let T be a plain ring with unit, T_n a total matrix ring of degree n over T and U a nil-subring of T_n. In case T is a division algebra it has been shown by the author[11] that U must be nilpotent and that $U^n = 0$. In the present section we shall show that this is true also in case T is an arbitrary plain ring. This will follow from the following theorem:

THEOREM 5. *Let* S *be a semi-simple* I-ring of finite index* i(S)=r. *Then every nil-subring* U *of S is nilpotent and* $U^r = 0$.

Proof. Since U is of finite index it follows (compare [12]) that U is semi-nilpotent, i.e. for an arbitrary integer k, and in particular for $k = r$, any set of r elements $u_1, u_2, \ldots, u_r$ in U generates a nilpotent ring $V = \{u_1, u_2, \ldots u_r\}$. Hence, for some integer m we have

$$V^{m-1} \neq 0, \quad V^m = 0 \tag{3}$$

We choose the $m-1$ elements $v_i \in U$, $i = 1, \ldots, m-1$ so that $v = v_1 v_2 \ldots v_{m-1} \neq 0$. Since S is a semi-simple I-ring, the right ideal vS is not nil and contains a nonzero idempotent e. Thus for some $s \in S$ we have $vs = v_1 v_2 \ldots v_{m-1} s = e$. Consider now the $m-1$ elements

$$w_1 = v_1 \ldots v_{m-1} s v_1, \quad w_2 = v_2 \ldots v_{m-1} s v_1 v_2, \quad \ldots, \quad w_{m-1} = v_{m-1} s v_1 \ldots v_{m-1} \tag{4}$$

and put $w = \sum_{i=1}^{m-1} w_i$. By (3) we know that the product of any m of the v's is $= 0$. Hence by (4) and by the definition of w one easily verifies that $w^m = 0$ while $w^{m-1} = (v_1 v_2 \ldots v_{m-1} s)^{m-1} v_1 \cdot v_2 \ldots v_{m-1} = e v_1 v_2 \ldots v_{m-1} \neq 0$, i.e. the element w is nilpotent of index m. By the assumption of the theorem this yields $m \leqslant r$. Thus in view of (3) we have $V^r = 0$, and as $u_i \in V$ we obtain in particular $u_1 . u_2 \ldots u_r = 0$. Since the u's were chosen arbitrarily in U this shows that $U^r = 0$, q. e. d.

A total matrix ring T of degree n over a plain ring T is a semi-simple* I-ring (compare the first part of Lemma 3.1 in ref. 13). Hence we obtain in view of Theorem 4 as a special case of Theorem 5 the following result:

Corollary. If U is a nil-subring of T_n, then U is nilpotent and $U^n = 0$.

Remark. In a recent paper Nagata[15] has proved that if U is a nil-algebra of finite index m, over a field Φ of characteristic zero, then U is nilpotent, and the index of nilpotency of U depends only on m. He also constructed a simple example of a non nilpotent nil-algebra of finite index, over a field Φ with a nonzero characteristic. Since any nil-algebra is an I-ring, Nagata's example shows that in our Theorem 5 the assumption that S should be semi-simple is indispensable.

IV. A THEOREM ON SEMI-SIMPLE SUBRINGS OF T_n.

One readily verifies the validity of the following lemmas:

Lemma 3. *A two sided ideal of a total matrix ring of degree* n *over a plain ring* T *is also a total matrix ring of the same degree* n *over some plain subring* T′ *of* T.

* See footnote ** on page 1.

Lemma 4. *If* S *is a semi-simple I-ring and* e *an idempotent in* S, *then the subring* eSe *is also a semi-simple I-ring.*

We shall need also Lemma 2.3 of ref.13 which we restate here without proof:

Lemma 5. *If a semi-simple I-ring* S *contains a complete system of* n^2 *matrix units* e_{ik}, $i,k = 1, \ldots, n$ *such that either the ring* $e_{11}Se_{11}$ *is not plain, or the ring* $e_{11}Se_{11}$ *is plain but the element* $e = \sum_{i=1}^{n} e_{ii}$ *is not in the centre, then there exists a complete system of* $(n+1)^2$ *matrix units* $d_{ik}, i,k = 1, \ldots, n+1$, *such that* $d_{11} \in e_{11}Se_{11}$.

We shall use the foregoing three lemmas in the proof of the following theorem:

THEOREM 6. *Let* $S = T_n$ *and* $S' = T'_{n'}$ *be total matrix rings of degrees* n *and* n′ *over the plain rings* T *and* T′ *respectively. Suppose further that* $S' \subset S$ *and that* S *has the identity element* e *in common with* S′. *Then* n′ *divides* n.

Proof. By assumption we have $S' = \sum_{i,k=1}^{n'} e'_{ik} T'$, where the e'_{ik} constitute a complete basic matrix units system in S'. Every element e'_{ik} commutes with every element of T', and the element $e = \sum_{i=1}^{n'} e'_{ii}$ is the identity of S and of S'. Consider the rings $S_{ik} = e'_{i1} S e'_{1k}$. One verifies readily that

$$e'_{ij} S e'_{rk} = S_{ik} \text{ for any indices } i, j, r, k \qquad (5)$$

By Theorem 4 we have $i(S) = n$, hence by $i(S_{11}) \leqslant i(S)$ it follows that $i(S_{11}) \leqslant n$. Put $m = i(S_{11})$. Since S is a semi-simple I-ring we know by Lemma 4 that S_{11} is also a semi-simple I-ring. In view of Theorem 1 the ring S_{11} contains a complete system of m^2 matrix units d_{ik}, $i,k = 1,\ldots,m$ and by Lemma 2 and Lemma 5 there are no complete matrix units systems of r^2 matrix units in S_{11} if $r > m$. Hence it follows by Lemma 5 that the idempotent $d_1 = \sum_{i=1}^{m} d_{ii}$ lies in the centre of S_{11}. We wish to show that the idempotent $d = \sum_{i=1}^{n'} e'_{i1} d_1 e'_{1i}$ lies in the centre of S. For $s \in S$ we have $s = ese = \sum_{i,k=1}^{n'} e'_{ii} s e'_{kk}$. By (5) we have $e'_{ii} s e'_{kk} \in e'_{i1} S_{11} e'_{1k}$ so that we may put $e'_{ii} s e'_{kk} = e'_{i1} s_{ik} e'_{1k}$, where $s_{ik} \in S_{11}$, and we have $s = \sum_{i,k=1}^{n'} e'_{i1} s_{ik} e'_{1k}$. Since d_1 lies in the centre of S_{11} we have $d s_{ik} = s_{ik} d$, $i,k = 1, \ldots, n'$ which easily implies that $ds = sd$, so that indeed d lies in the centre of S. Consider now the two sided ideal $dS = Sd$ of S. This ideal contains a complete system of $n'm$ matrix units f_{ik} which are defined as follows:

$$f_{(i-1)t+j,(r-1)t+k} = e'_{i1} d_j e'_{1r}, \quad i, r = 1, \ldots, n'; \; j,k = 1, \ldots, m$$

By Lemma 5 the ring $d_{11}Sd_{11}$ $(= d_{11}S_{11}d_{11})$ is plain. For any $s \in S$ the correspondence $d_{11}sd_{11} \to \sum_{i=1}^{n'm} d_{i1} s d_{1i}$ is an isomorphism. Hence the set T^* of all the elements of the form $\sum_{i=1}^{n'm} d_{i1} s d_{1i}, s \in S$ is a plain ring. Since evidently $dS = \sum_{i,k=1}^{m} d_{ik} T^*$ where the d_{ik} commute

with every element of T^*, it follows that dS is a total matrix ring of degree mn' over the plain ring T. On the other hand, since dS is a two sided ideal in T_n, it must be a total matrix ring of degree n (compare Lemma 3). By the invariance of the degree of a total matrix ring over a plain ring (compare Theorem 3) it follows therefore that $n=mn'$. i.e. n' divides n, q. e. d.

Remark. In the special case where T and T' are division rings the above theorem was proved in ref. 10, Satz 1, Folgerung 2. The case where T is strongly regular, i.e. a plain FI-ring (compare ref. 13, Theorem 5.5), we can reduce the proof to the case of division algebras by passage to the primitive images of S. However, this device is not available in the general case, and the roundabout manner of the above proof which compelled us to apply Theorems 1, 2, 3 and 5 seems to be unavoidable.

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ON THE SINGULARITIES OF A CLASS OF DIRICHLET SERIES

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We shall consider in this paper Dirichlet series of the type

$$f(s)=\sum_{n=0}^{\infty} a_n e^{-\lambda_n s} \tag{1}$$

where $0\leqslant \lambda_n \uparrow \infty$ and $\liminf (\lambda_{n+1}-\lambda_n)=h>0$.

It is well known that these series have many of the characteristics of the Taylor-D series:

$$f(s)=\sum_{n=0}^{\infty} a_n e^{-ns}.$$

Thus, the abscissas of convergence, absolute convergence and holomorphy of (1) coincide. Also, denoting by D_λ the maximal density of the sequence $\{\lambda_n\}$ in the sense of Pólya:

$$D_\lambda = \lim_{\xi=1} \limsup_{r=\infty} [N(r)-N(r\xi)]/(r-r\xi), \tag{2}$$

($N(r)=\Sigma_{\lambda_n<r} 1$ and $0<\xi<1$) it is well known (Pólya) that every segment of the axis of convergence of length greater than $2\pi D_\lambda$ contains at least one singular point of $f(s)$. This last result indicates that something of the periodic character of the singularities of the Taylor-D series is retained in the case of series (1). We propose in this note to develop further this analogy by proving the following theorem:

THEOREM: *Let* f(s) *be an analytic function represented in the half-plane of convergence*: $\mathrm{Re}\{s\}>\sigma_c>-\infty$ *by* (1), *and let* D_λ *be the maximal density* (2). *Suppose that the only singularities of* f(s) *on a segment* I *of the axis of convergence of length greater than* $2\pi(D_\lambda+1/h)$ *are the points* $\sigma_c+i\alpha_q$, $q=1,\ldots,k$, *which are simple poles. Denote by* $S_f^{\sigma_c}$ *the set of singularities of* f(s) *on the axis of convergence. Then, there exists* $\delta>0$ *such that* f(s) *is analytic and uniform in the domain consisting of the points of the half-plane* $\mathrm{Re}\{s\}>\sigma_c-\delta$ *not belonging to* $S_f^{\sigma_c}$. *Also, every isolated singularity of* $S_f^{\sigma_c}$ *is a simple pole of* f(s). *Furthermore, if* $\sigma_c+i\alpha$ *is such a point then*:

$$\alpha=m_1\alpha_1+m_2\alpha_2+\ldots+m_k\alpha_k, \tag{3}$$

where the m's *are integers.*

The theorem thus states that the singularities of $f(s)$ in I determine in a very specific way the character of all other singularities on the axis of convergence (and even in a half-plane containing the axis of convergence). The last property (3) shows further that though we do not have here periodicity of singularities we have something similar in the fact that $\alpha_1,\ldots,\alpha_k$, constitute an integral base of the imaginary part of all other isolated singularities on the axis of convergence.

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Before proving the theorem we mention that the number $2\pi\ (D_\lambda + 1/h)$ of the statement can be replaced by $4\pi D_\lambda$, and that one could also establish that $f(s)$ is at most of order 1 "except for singularities" in the half-plane $Re\{s\} > \sigma_c - \delta$. It is also possible to establish similar results when the poles are not assumed to be simple. The more general results, however, necessitate the extension of the method used here to series of the type

$$f(s) = \sum p_n(s)\, e^{-\lambda_n s}$$

where the $p_n(s)$ are polynomials.

Proof of the Theorem. We start by noting that there is no loss of generality in assuming the imaginary axis to be the axis of convergence (i.e. $\sigma_c = 0$). In the proof we shall use the method of principal indices [1 ; 2]* and we shall first recall some results connected with this method which are applicable in this case. From the assumptions on the character of singularities of $f(s)$ it follows [2; p. 467, (c)] that $\lambda_{n+1} - \lambda_n$ is bounded and that there exists a constant $C > 0$ and an integer $N > 0$, such that

$$1/C \leqslant \max_{\nu=1,\ldots,N} |a_{n+\nu}| \leqslant C, \tag{4}$$

for all n sufficiently large. Thus, in the terminology of [1; p. 271, Def. C] the sequence of coefficients $\{a_n\}$ is quasi-regular. For $0 \leqslant x < \infty$ we set

$$f_x(s) = [f(s) - \sum_{\lambda_n < x} a_n e^{-\lambda_n s}]/e^{-xs} \tag{5}$$

$$= \sum_{\lambda_n \geqq x} a_n e^{-(\lambda_n - x)s} \qquad \text{for } Re\{s\} > 0.$$

Hence, $f_x(s)$ is analytic in any closed domain Δ where $f(s)$ is analytic. Moreover, by the Fundamental Theorem of [1; 2], the family $\{f_x(s)\}$ is uniformly bounded in Δ and its limit functions $g(s)$:

$$g(s) = \lim_{x_j = \infty} f_{x_j}(s), \tag{6}$$

possess the following properties:

(a) $g(s)$ is analytic and uniform in the whole plane punctured by the set of singularities of $f(s)$ on the axis of convergence (i.e. the set S_f^0).

(b) It possesses developments:

$$g(s) = \sum_{n=0}^{\infty} b_n e^{-\mu_n s} \qquad \text{for } Re\{s\} > 0,$$

$$g(s) = -\sum_{n=1}^{\infty} b_{-n} e^{\mu_{-n} s} \qquad \text{for } Re\{s\} < 0,$$

where $\mu_{\pm n} \geqq 0$, $\mu_{\pm(n+1)} - \mu_{\pm n} \geqq h$, and $|b_{\pm n}| \leqslant C$.

(c) If $i\alpha$ is an isolated point of S_f^0 which is also a singularity of $g(s)$, then it is a simple pole of $g(s)$.

* Numbers in brackets refer to the references cited at the end of the paper.

(d) $g(s)$ is non-constant and possesses singularities on the axis of convergence. (This results from the quasi-regularity of $\{a_n\}$.)

We shall now verify the last property (d) directly by showing that all simple poles $i\alpha$ of $f(s)$ on the imaginary axis are also simple poles of $g(s)$. Indeed, let $i\alpha$ be such a pole:

$$f(s) = r/(s-i\alpha) + \text{regular function} \qquad (7)$$

in a certain disk $|s-i\alpha| \leqslant \delta$. Let $g(s)$ be the limit function given by (6). Then, we will show that in the same disk

$$g(s) = r'/(s-i\alpha) + \text{regular function},$$

where

$$r' = r \lim_{x_j=\infty} e^{i\alpha x_j}. \qquad (8)$$

Indeed, $\{f_{x_j}(s)\}$ is analytic in the disk except for a simple pole at $s=i\alpha$ with residue $re^{i\alpha x_j}$. Also $\{f_{x_j}(s)\}$ tends uniformly to $g(s)$ on the circumference. Setting:

$$\varphi_{x_j}(s) = f_{x_j}(s) - re^{i\alpha x_j}/(s-i\alpha), \qquad (9)$$

and choosing a subsequence $\{j'\}$ of the integers so that $e^{i\alpha x_{j'}}$ tends to a limit, we find that $\varphi_{x_{j'}}(s)$ is analytic in the disk and tends uniformly to a limit on its boundary; hence, it tends to an analytic function in its interior as well. Thus, we conclude (using (6) and (9)) that $g(s) = r'/(s-i\alpha)$ + reg. function in the disk where $r' = r \lim \exp(i\alpha x_{j'})$ $(j' \to \infty)$. However, $\exp(i\alpha x_j)$ $(j \to \infty)$ cannot possess more than one limit point (since this will yield another value for the residue r' at $i\alpha$) and hence (8) holds.

Now, there exists a limit function $g(s)$ having the same singularities as $f(s)$ on the segment I of the imaginary axis. Indeed, by assumption $f(s)$ possesses only k simple poles $i\alpha_q$ in I. Let r_q be the residue at $i\alpha_q$. Then, we have just shown that any limit function will have the same simple poles in I with respective residues r'_q. Furthermore, if $g(s)$ is the limit function of (6) then the relation between r_q and r'_q was shown to be:

$$r'_q = r_q \lim_{x_j=\infty} e^{i\alpha_q x_j}, \qquad q = 1, \ldots, k.$$

We shall, therefore, obtain the desired limit function if we could choose the sequence $\{x_j\}$ $(x_j \to \infty)$, so that, for $q=1, \ldots, k$, $\lim \exp(i\alpha_q x_j) = 1$ (equivalently, $\alpha_q x_j \pmod{2\pi}$ tends to zero). But, the possibility of choosing a sequence $\{x_j\}$ with the above properties follows readily from Dirichlet's approximation theorem. Hence, the existence of such a limit function $g(s)$ is assured.

Let $F(s)=f(s)-g(s)$ where the limit function $g(s)$ was chosen so that $F(s)$ is regular in I. In the half-plane $Re\{s\}>0$ $F(s)$ will possess the Dirichlet series representation:

$$F(s) = \sum a_n e^{-\lambda_n s} - \sum b_n e^{-\mu_n s} = \sum A_n e^{-\Lambda_n s}, \qquad (10)$$

where $\{\Lambda_n\}$ is the union sequence of both sequences $\{\lambda_n\}$ and $\{\mu_n\}$, arranged in

the natural order. Now, it follows immediately from (2), if D_Λ, D_λ and D_μ denote the maximal densities of $\{\Lambda_n\}$, $\{\lambda_n\}$ and $\{\mu_n\}$ respectively, that

$$D_\Lambda \leqslant D_\lambda + D_\mu \leqslant D_\lambda + 1/h,$$

since $\mu_{n+1} - \mu_n \geqq h$. Now, by a V. Bernstein [3, p. 138] generalization of a theorem of Pólya it follows that every segment of the axis of holomorphy of (10) of length greater than $2\pi D_\Lambda$ contains at least one singularity of $F(s)$. Since, however, $F(s)$ is analytic on the segment I of the imaginary axis whose length is greater than $2\pi D_\Lambda$, it follows that the imaginary axis is not the axis of holomorphy of $F(s)$ and that there exists $\delta > 0$ such that $F(s)$ is analytic in the half-plane $Re\{s\} > -\delta$. Hence, in this half-plane we may write

$$f(s) = g(s) + \text{reg. function.}$$

But, $g(s)$ being a limit function it follows from the properties of such functions mentioned above that $f(s)$ is analytic and uniform in the whole plane $Re\{s\} > -\delta$ punctured by the set S_f^0, and that all isolated points of S_f^0 are simple poles of $f(s)$. This proves the first part of the theorem.

It is left only to show that if $i\alpha$ is an isolated point of S_f^0 (a simple pole by the above result) then:

$$\alpha = m_1\alpha_1 + m_2\alpha_2 + \ldots, + m_k\alpha_k, \tag{11}$$

where $m_1, m_2, \ldots, m_k$ are integers. For this purpose we need the following approximation theorem: Let $\alpha, \alpha_1, \ldots, \alpha_k$ be $k+1$ real numbers, and suppose that α is not of the form (11) with integral m's. Then, there exists $0 < \eta_0 < 1/2$ such that for each $\varepsilon > 0$ and $T > 0$, one can find a real number $t > T$ and integers $n, n_1, \ldots, n_k$ satisfying:

$$|t\alpha_j - n_j| < \varepsilon \text{ for } j=1, \ldots, k, \text{ and } \eta_0 < |t\alpha - n| < 1-\eta_0. \tag{12}$$

Assuming for a moment the truth of this result we shall establish (11). Indeed, assume that the singularity $i\alpha$ is such that α is not of the form (11). It follows readily that with a given sequence $\{\varepsilon_j\}$, $\varepsilon_j \downarrow 0$ one can associate a sequence $\{x_j\}, (x_j \to \infty)$ and $k+1$ sequences of integers $\{n^j\}$, $\{n_1^j\}, \ldots, \{n_k^j\}$, such that

$$|x_j\alpha_q - 2\pi n_q^j| < \varepsilon_j \text{ for } q=1, \ldots, k, \text{ and } \eta_0 2\pi < |x_j\alpha - 2\pi n^j| < (1-\eta_0)2\pi \tag{12'}$$
$$\text{for } j=1, 2, \ldots.$$

Choosing $g(s)$ as a limit function of the family $\{f_{x_j}(s)\}$ with $\{x_j\}$ satisfying (12'), one finds as before that $g(s)$ has the same residues as $f(s)$ at the points $i\alpha_q$ $(q=1, \ldots, k)$ and the previous argument leads to the conclusion that $f(s) - g(s)$ is analytic on the imaginary axis. However, the residue r' of $g(s)$ at the point $i\alpha$ differs from the residue r of $f(s)$ at the same point, since by (8) and (12')

$$r' = r \lim_{x_{j'} = \infty} e^{i\alpha x_{j'}} = r e^{i\theta 2\pi} \neq r \quad (\text{since } \eta_0 \leqslant \theta \leqslant 1 - \eta_0). \tag{B}$$

This leads to a contradiction and proves (11).

We still have to prove the above approximation theorem. But, this is an easy consequence of Kronecker's general approximation theorem [4, p. 83]. Of Kronecker's general theorem we need only the following special case:

APPROXIMATION THEOREM. *Let* $\varepsilon > 0$ *and* $T > 0$ *be given. Then, the system of inequalities*

$$\begin{aligned} &-\varepsilon < t\alpha - \beta < \varepsilon \pmod 1 \\ &-\varepsilon < t\alpha_i < \varepsilon \pmod 1 \text{ for } i = 1, \ldots, k, \end{aligned} \tag{13}$$

admits a solution $t > T$ *if the following is true: whenever* $n, n_1 \ldots, n_k$, *are integers such that* $n\alpha + n_1\alpha_1 + \ldots + n_k\alpha_k = 0$, *then* $n\beta$ *is an integer.*

Now, if α is not a linear combination with integral coefficients of $\alpha_1, \ldots, \alpha_k$ it is always possible to choose a number β, $0 < \beta < 1$, for which the condition of the last theorem will hold so that (13) could be satisfied for arbitrary ε and T. (This, of course, will include the previously used result (12)). Indeed, if α is not even a linear combination with rational coefficients of $\{\alpha_i\}$ $(i = 1, \ldots, k)$ then it is obvious that the condition of the Approximation Theorem is satisfied for any β, $0 < \beta < 1$. Suppose now that α is a linear combination with rational coefficients of $\alpha_1, \ldots, \alpha_k$, then there exist integers $m > 1$, $m_1, \ldots, m_k$, such that:

$$m\alpha = m_1\alpha_1 + \ldots + m_k\alpha_k.$$

If $\mu > 1$ is the smallest m with the above representation it is readily seen that any other m such that $m\alpha$ is a linear combination with integral coefficients of $\{\alpha_i\}$ is a multiple of μ. Choosing $\beta = 1/\mu$ $(0 < \beta < 1)$ one sees easily that the condition of the Approximation Theorem is satisfied so that (13) holds for a suitable t. As we have already noted, (12) follows from (13) and thus our proof is complete.

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INDEFINITE ADMITTANCE REPRESENTATION OF LINEAR NETWORK ELEMENTS*

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INTRODUCTION

The various methods of network analysis — analysis by meshes, branches or nodes — are based on two postulates, known as Kirchhoff's Laws. These postulates concern the currents in branches meeting at any node, and the voltages across branches forming the contour of any closed mesh; it is thus implied that the network is composed of two-terminal branches, interconnected into a configuration that is called (in topology) a "graph". The two following postulates are implied about each branch:

1. It is possible to define a current *through* the branch; or, the current entering at one terminal is equal to that leaving at the other one.

2. The current through the branch depends only on the voltage *difference* between both terminals, and not on their absolute potentials.

When elements that have more than two terminals, like vacuum-tubes or transistors, are incorporated in the network, the usual practice is to represent the element by an equivalent network of two-terminal branches and "internal" sources, so that the branch postulates and Kirchhoff's Laws may be applied.

A different approach is outlined in this paper. Each network element is characterised by the number of its terminals, n. The only currents used in the analysis are those entering the terminals, and the only voltages considered are those between terminals. No "internal" structure of the element is implied other than a possible partitioning of the set of n terminals into s subsets, with the restriction that an external source may be connected, or a voltage difference measured, only between terminals belonging to the same subset. It is immaterial whether this partitioning is due to the physical structure of the element (as in the four-terminal element composed of two coils linked by mutual inductance only) or to some special application (e.g., the subsets of input and output terminals in a four-pole or a multi-terminal transducer). If the terminals are partitioned into subsets, their numbering will be such that the terminals of a subset will have consecutive numbers.

Throughout the paper, the term "subset" will refer to the special subset described above. The terminals of a typical subset will be numbered from p to q consecutively.

Two postulates, concerning the currents into the terminals and the voltages at the terminals of any subset, are proposed as a basis for element representation. These postulates replace 1 and 2 above, which apply only to the special case $n=2$, $s=1$. The voltages are not completely defined, because the voltage reference terminal is left un-

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specified. It will be shown that this indefiniteness of the voltages causes no ambiguity in the definitions and evaluations of power and admittance.

Current

Let the current entering the k-th terminal from an external source be denoted by i_k; it may be regarded as the k-th component of an n-rowed column matrix $\mathbf{i}$. This matrix may be partitioned into s subcolumns, according to the partitioning of the terminals into s subsets.

The first postulate concerns the elements of each subcolumn of $\mathbf{i}$.

P1. *The sum of all the currents entering the terminals of any subset is zero.*

$$\sum_{k=p}^{q} i_k = 0. \tag{1}$$

The n components of $\mathbf{i}$ are therefore interdependent, and have only n-s degrees of freedom.

Voltage

Let a voltage be ascribed to each terminal, with all voltages v_j forming an n-rowed column matrix $\mathbf{v}$, similarly partitioned into s subcolumns. It is not specified which terminal is to be used as a voltage reference terminal in each subset, but it is assumed that the currents $\mathbf{i}$ are determined only by voltage differences between terminals belonging to the same subset. This is expressed by the second postulate.

P2. *An arbitrary voltage added to the voltages of all the terminals of any subset has no effect on the current into any terminal.*

$$\begin{aligned} &i_k(v_1,\cdots,v_{p-1},v_p,v_{p+1},\cdots,v_{q-1}\ v_q,v_{q+1},\cdots,v_n) = \\ &= i_k(v_1,\cdots,v_{p-1},v_p+v_0,v_{p+1}+v_0,\cdots,v_{q-1}+v_0,v_q+v_0,v_{q+1},\cdots,v_n) \end{aligned} \tag{2}$$

for an arbitrary v_0.

This indefiniteness of the voltages enables a generalised analysis, without recourse to specified voltage reference terminals ("ground") as in other methods of nodal analysis.

Power

The currents and voltages, in the general case, are functions of time, and the instantaneous power supplied to the element is

$$P = \sum_{k=1}^{n} i_k v_k = \mathbf{i}'\mathbf{v} \tag{3}$$

where $\mathbf{i}'$ is the transpose of $\mathbf{i}$. This expression is unaffected by the indefiniteness of the voltages; assume that an arbitrary v_0 be added to all the voltages of one subset, then the new value of instantaneous power will be

$$P = \sum_{k=1}^{p-1} i_k v_k + \sum_{k=p}^{q} i_k (v_k+v_0) + \sum_{k=q+1}^{n} i_k v_k = \sum_{k=1}^{n} i_k v_k + \Big[\sum_{k=p}^{q} i_k\Big] v_0 \ ,$$

and this equal to (3), because the added term is zero (postulate P1).

Admittance

The elements of the column **i** have less degreees of freedom than those of **v**, so that it is possible to express the relation between them by

$$\mathbf{i} = \mathbf{Y}\mathbf{v} \tag{4}$$

where **Y** is a singular operator. In linear network elements, **Y** is a square matrix of order n, whose elements are linear operators. The **Y** matrix has a special structure, which is a result of P1 and P2.

Take the expression for any current,

$$i_k = \sum_{j=1}^{n} Y_{kj} v_j \tag{5}$$

Summing all the currents into any subset of terminals,

$$\sum_{k=p}^{q} i_k = \sum_{k=p}^{q} \sum_{j=1}^{n} Y_{kj} v_j = \sum_{j=1}^{n} \left[\sum_{k=p}^{q} Y_{kj} \right] v_j \tag{6}$$

but according to P1, this sum is zero for any **v**, so that

$$\sum_{k=p}^{q} Y_{kj} = 0 \tag{7}$$

Now suppose that an arbitrary v_0 is added to all the voltages of any subset, then (5) will give.

$$i_k = \sum_{j=1}^{p-1} Y_{kj} v_j + \sum_{j=p}^{q} Y_{kj} (v_j + v_0) + \sum_{j=q+1}^{n} Y_{kj} v_j = \sum_{j=1}^{n} Y_{kj} v_j + \left[\sum_{j=p}^{q} Y_{kj} \right] v_0 \tag{8}$$

Comparing (5) and (8), by postulate P2

$$\left[\sum_{j=p}^{q} Y_{kj} \right] v_0 = 0$$

and as v_o is arbitrary,

$$\sum_{j=p}^{q} Y_{kj} = 0 \tag{9}$$

Equations (7) and (9) describe the structure of **Y**: let **Y** be partitioned, both horizontally and vertically, according to the subsets of terminals, then each one of the resulting s^2 submatrices has the property that the sum of the elements in each row and each column is zero.

Evidently, the determinant of **Y** is zero, and so will be any subdeterminant as long as it contains all the rows or all the columns of any subset. Only if at least one row and one column are left out of every subset, the determinant may be different from zero. The *nullity* of the admittance matrix is at least s.

A special case of some interest is when a subset contains only one terminal. (The terminal in this case is an "isolated" terminal, because no current can enter into it). In this case **Y** has a row and column of zeroes. Any m-terminal network element may

thus be treated as having n terminals ($n > m$), if its admittance matrix of order m is augmented by adding $n-m$ rows and columns of zeroes.

The admittance matrix, whose structure is described by (7) and (9), that relates the indefinite voltages (P2) to the redundant currents (P1), will be termed the *indefinite admittance matrix* of the linear network element.

GEOMETRICAL INTERPRETATION

The n-rowed column matrices $\mathbf{v}$ and $\mathbf{i}$ may be represented in n-dimensional Euclidian space by points whose coordinates are $x_j = v_j$ and $x_k = i_k$ respectively. We shall first describe the case of a 3-terminal network element, whose terminals belong to a single subset.

Let $x_1x_2x_3$ be an Euclidian reference frame. Any point representing current is constrained by P1 to the plane

$$x_1 + x_2 + x_3 = 0, \tag{10}$$

which will be called the *current plane*. When any terminal is specified as a voltage reference terminal, only two currents and voltages are needed for the analysis. Suppose terminal 3 to be chosen for reference, then the *definite* representation is in the x_1x_2 plane, and the two components of the current that are used in the analysis are represented by the projection on this plane of the point in the current plane.

The line

$$x_1 = x_2 = x_3 \tag{11}$$

passes through the origin and is perpendicular to the current plane. Any voltage represented by a point on this line corresponds to zero current; this line will therefore be called the *null line*.

Take now any point representing a given set of terminal voltages, and through it draw a straight line parallel to the null line. P2 states that all the voltages represented by points on this line correspond to a single current point. If terminal 3 is chosen for voltage reference, the definite voltage is the point where this voltage line intersects the x_1x_2 plane, for there $x_3 = 0$.

Instantaneous power, defined by (3), is the inner product of two vectors; the current vector from the origin to the current point, and the voltage vector from the origin to *any* point on the voltage line. But this voltage line is perpendicular to the current plane, so that any two voltages differ only by a vector normal to any current vector; therefore the inner product is independent of the point on the line chosen for voltage representation.

The indefinite admittance matrix of order 3 establishes a correspondence between *any* point on the voltage *line* to a *single* point on the current plane, so it must be a singular matrix.

This description is easily extended to a network element having n terminals in s subsets. In the indefinite analysis, voltages and currents are represented by points in n-dimensional space; when s terminals, one in each subset, are specified as voltage reference terminals, the definite current and voltage points are in a subspace of $n-s$ dimensions, the *reference subspace*.

P1 constrains all the current points to a *current subspace* of $n-s$ dimensions, which is defined by s equations of the type

$$x_p + x_{p+1} + \ldots + x_{q-1} + x_q = 0. \qquad (12)$$

The *null space* of s dimensions, defined by s systems of equations of the type

$$x_p = x_{p+1} = \ldots = x_{q-1} = x_q \qquad (13)$$

is the orthogonal complement of the current subspace. An indefinite voltage is represented by an s-dimensional space "parallel" to the null space, and therefore orthogonal to any vector in the current subspace, which accounts for the unambiguous expression for power.

In the definite representation, the currents that are used in the analysis are represented by the projection of the indefinite current point onto the reference subspace; the definite voltage is the intersection of the s-dimensional voltage space with the $(n-s)$-dimensional reference subspace.

An indefinite admittance matrix establishes a correspondence between any point in an s-dimensional voltage space to a single current point in the current subspace, therefore its nullity is at least s. In effect, the current subspace (12) is the *row space* and *column space* of any indefinite admittance matrix, i.e., the space spanned by the vectors that form the rows and columns of the matrix, as is evident from (7) and (9); the *null space* of any such matrix is given by (13).

CONCLUSION

There are two fundamental problems that underlie any method of network analysis: first, how each network element is to be represented, and second, how to represent the interconnections of the elements that form the network. In networks composed of two-terminal branches only, the element representation is based on the two postulates mentioned in the Introduction, and the interconnection of elements in a network is subjected to Kirchhoff's Laws.

The representation of multi-terminal network elements has been treated in this paper. The interconnection of such elements into a network cannot be based on Kirchhoff's laws as stated for networks of branches. The need for revision is especially obvious in the Mesh Law, for meshes can be defined only in a network with two-terminal elements. The representation of interconnections, and the postulates on which it is based, are outside the scope of this paper.

The indefinite admittance representation of network elements was applied by the author in a number of articles that appeared in the literature[1,2,3,4]. These articles treat some 3-terminal network elements (vacuum-tubes, transistors and gyrators), as well as multi-terminal transducers (elements with $2n$- terminals in two subgroups of n terminals).

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THE APPLICATION OF THE DEBYE CHARGING PROCESS TO POLYIONS

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In a recent paper[1] we defined the polyionic work of a polyion as the electrical work required to build up the polyion from the corresponding single ions. Using the Debye limiting law and a Güntelberg charging process we obtained a general expression for the polyionic work of a rigid polyion in a salt solution of given concentration. The object of the present paper is to show how this same expression can be deduced using a Debye charging process.

Consider two electrically neutral systems, the first consisting of a single rigid polyion carrying ν charges of magnitude ε present in an aqueous solution of volume V together with a very large number N of single ions carrying charges $\pm\varepsilon$, and the second consisting of an aqueous solution having the same volume V but containing $(N+\nu)$ single ions carrying charges $\pm\varepsilon$. The polyionic work will be equal to the difference in the electrical work required to charge the two systems by a Debye charging process. When evaluating the electrical work, we shall not include that part corresponding to the self potential of the charges, since this will be the same for both systems.

In the case of the first system, when the degree of charging is λ, it follows from the Debye limiting law that the potential on the Ith charge of the polyion, due to all the other polyion charges and to the ionic atmosphere, will be

$$\sum_{J\neq I} \frac{\lambda\varepsilon}{Dl_{IJ}} e^{-\lambda\varkappa l_{IJ}} - \frac{\lambda^2\varepsilon\varkappa}{D}$$

where l_{IJ} is the distance between the Ith and Jth polyion charges and $\varkappa$ is the usual Debye parameter.

The corresponding part of the electrical work required to charge all the ν charges of the polyion will therefore be

$$\sum_{I}\sum_{J\neq I}\int_0^1 \frac{\lambda\varepsilon^2}{Dl_{IJ}} e^{-\lambda\varkappa l_{IJ}} d\lambda - \frac{\nu\varepsilon^2\varkappa}{3D}. \qquad (1)$$

According to the Debye-Hückel theory, a volume element $d\tau$, having a potential ψ due to the charges on the polyion and its ionic atmosphere, will contain $(N/2V)(1 \mp \lambda\varepsilon\psi/kT)\,d\tau$ single ions having charges $\pm\lambda\varepsilon$ respectively, the potentials on these ions due to the polyion charges and the ionic atmosphere being $(\psi \mp \lambda^2\varepsilon\varkappa/D)$ respectively. The corresponding part of the electrical work required to charge the N single ions present will therefore be

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$$(N\varepsilon/2V)\int_0^1\int_V \left[(1-\lambda\varepsilon\psi/kT)(\psi-\lambda^2\varepsilon\varkappa/D)-(1+\lambda\varepsilon\psi/kT)(\psi+\lambda^2\varepsilon\varkappa/D)\right] d\tau\, d\lambda$$
$$= -(N\varepsilon^2/VkT)\int_0^1\int_V \lambda\psi^2 d\tau\, d\lambda - N\varepsilon^2\varkappa/3D. \qquad (2)$$

If the distance of the volume element $d\tau$ from the Ith polyion charge is denoted by l_I, then the potential ψ will be given by

$$\psi = \sum_I \frac{\lambda\varepsilon}{Dl_I} e^{-\lambda\varkappa l_I}$$

so that expression (2) takes the form

$$-\frac{N\varepsilon^4}{VD^2kT}\left[\sum_I \int_0^1\int_V \frac{\lambda^3}{l_I^2} e^{-2\lambda\varkappa l_I}\, d\tau d\lambda + \sum_I\sum_{J\neq I}\int_0^1\int_V \frac{\lambda^3}{l_I l_J} e^{-\lambda\varkappa(l_I+l_J)} d\tau d\lambda\right] - \frac{N\varepsilon^2\varkappa}{3D}. \qquad (3)$$

Now the Debye parameter $\varkappa$ is defined by $\varkappa^2 = 4\pi\varepsilon^2 c/DkT$, where c is the concentration of the (monovalent) ions actually forming the ionic atmosphere. In general it will be both convenient and sufficiently accurate to put $c = N/V$, but, owing to its large magnitude, the last term in expression (3) must be evaluated more accurately. Since this term corresponds to the potential due to the ionic atmosphere of a single ion formed by the remaining $(N-1)$ single ions, the appropriate value for c will be $(N-1)/V$. The first set of integrals occurring in the above expression can be evaluated, using a volume element $d\tau$ having the form of a spherical shell with I as centre, while the second set can be evaluated using confocal elliptic coordinates, I and J being taken as the foci; furthermore, V is large enough to be taken as infinite in the evaluation of these integrals.

On carrying out the calculations and combining the result with (1) we see that the relevant part of the electrical work required to charge the first system is

$$-\frac{\nu\varepsilon^2\varkappa}{2D} + \frac{\varepsilon^2}{2D}\sum_I\sum_{J\neq I}\frac{1}{l_{IJ}}\int_0^1\left[2\lambda e^{-\lambda\varkappa l_{IJ}} - \varkappa l_{IJ}\lambda^2 e^{-\lambda\varkappa l_{IJ}}\right] d\lambda - \frac{N\varepsilon^2}{3D}\left(\frac{4\pi\varepsilon^2(N-1)}{DkTV}\right)^{1/2}.$$

In the limit, as $N\to\infty$, this becomes

$$\frac{\varepsilon^2}{2D}\sum_I\sum_{J\neq I}\frac{1}{l_{IJ}} e^{-\varkappa l_{IJ}} - \frac{\varepsilon^2}{6D}\left(\frac{4\pi\varepsilon^2 N}{DkTV}\right)^{1/2}(2N-1+3\nu).$$

According to the usual Debye charging process, the corresponding part of the electrical work required to charge the second system will be $-(N+\nu)\varepsilon^2\varkappa'/3D$ where $\varkappa'$, the Debye parameter for the second system, is given by $\varkappa'^2 = 4\pi\varepsilon^2(N+\nu-1)/DkTV$. In the limit, as $N\to\infty$, this electrical work will be

$$-\frac{\varepsilon^2}{6D}\left(\frac{4\pi\varepsilon^2 N}{DkTV}\right)^{1/2}(2N+3\nu-1).$$

The polyionic work will therefore be equal to

$$\frac{\varepsilon^2}{2D}\sum_I\sum_{J\neq I}\frac{1}{l_{IJ}} e^{-\varkappa l_{IJ}},$$

a result obtained previously by means of a Güntelberg charging process.

In a similar manner it can be shown that the polyionic work of a rigid polyampholyte, carrying charges $+\epsilon$ at the points I, J etc. and charges $-\epsilon$ at the points I′, J′ etc. will be given by

$$\frac{\epsilon^2}{2D}\left[\sum_{\mathrm{I}}\sum_{\mathrm{J}\neq\mathrm{I}}\frac{1}{l_{\mathrm{IJ}}}e^{-\varkappa l_{\mathrm{IJ}}}+\sum_{\mathrm{I}'}\sum_{\mathrm{J}'\neq\mathrm{I}'}\frac{1}{l_{\mathrm{I}'\mathrm{J}'}}e^{-\varkappa l_{\mathrm{I}'\mathrm{J}'}}-2\sum_{\mathrm{I}}\sum_{\mathrm{I}'}\frac{1}{l_{\mathrm{II}'}}e^{-\varkappa l_{\mathrm{II}'}}\right].$$

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ON ELECTRICAL CONDUCTIVITY IN DETONATION AND SHOCK WAVES, AND THE MEASUREMENT OF DETONATION AND SHOCK VELOCITIES

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INTRODUCTION

Among the physical properties which characterize explosions, such as pressure, temperature and velocities of detonation and of shock waves, velocity is the only quantity that can be measured with comparative ease. The measurement of velocities is therefore used extensively in the study of explosions both from the practical and the theoretical points of view.

The methods of measuring detonation velocities may be divided into photographic and electronic. The photographic methods[1] are in turn subdivided into several groups, such as streak photography and high speed motion photography. In the latter rates of some millions frames per second are attainable. Photographic methods naturally show only the light visible on the outside of the explosive. This light is often produced by shock waves in the air outside the charge; thus care must be exercised in the interpretation of the results.

In electronic methods the time by which the shock wave moves a certain distance is measured on an oscillograph or with an electronic chronograph, and the velocity is thereby calculated. The oscillograph has the advantage over the chronograph in its ability to measure several consecutive time intervals, while the latter can measure only one time interval at a time. The time resolution obtainable with commercial chronographs is of the order of 10^{-7} seconds. With oscillographs time resolution of 10^{-8} sec or better[2] is obtainable, but it is not usually necessary because of the limited accuracy with which the distances are measured. Electronic methods, in contrast with photographic methods, may be used for the measurement of velocities in cased charges or inside the charge.

In electronic methods the arrival of the detonation front at particular points is signalled to the measuring instrument by sensing probes which are fitted in or on the charge. The probes are destroyed by the explosion, of course, and it is therefore desirable that they should be cheap. A good probe should sense the arrival of the detonating front with little or no delay and be physically small so as to interfere as little as possible with the detonation. A single thin wire may serve as a probe; the explosion on reaching the wire cuts it and breaks any current it may carry. A faster probe is obtained by two parallel wires spaced slightly apart which start conducting when swept by the ionized gases[3,4]. Measurements of shock wave velocity in air close to the charge may also make use of conduction caused by ionization[5] although other methods are available too, such as the use of corona probes which detect variations in pressure[6], or the use of microwaves for the determination of the ionized front velocity by the Doppler effect of the reflected waves[7].

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Although the understanding of the mechanism of electrical conductivity in detonations and shock waves is of interest both theoretically and practically, for example in velocity measurements, only little is known on the subject[8]. This is probably the result of complications due to the transient nature of the phenomena and the experimental difficulties.

In this paper the electrical conductivity between two electrodes, on the explosive or in the air close to it, is investigated with a view to obtaining data pertaining to the design of probes. Subsequently a system was developed for recording short time intervals. Some measurements of detonation and shock wave velocities are described to illustrate the high accuracy which can be obtained by the system.

THEORETICAL

In investigating the mechanism of electrical conductivity in gases ionized by explosions, we start with general principles and later, by checking the results, decide which factors are important and which assumptions are justifiable.

The current flowing between two electrodes in an ionized medium is equal to the scalar product of the current density (**j**) and the cross sectional area (**s**).

$$I = \mathbf{j} \cdot \mathbf{s} \qquad (1)$$

In our experiments the current varied, when measured on the charge, between 1—100 amperes, depending on the electrodes, on the voltage and on the type of explosive. In air the current was smaller and greatly affected by the strength of the shock wave. Difficulties are encountered as soon as we come to define the conducting cross sectional area. Near the electrodes it may be taken approximately as of the same order of magnitude as the area of the electrodes. In most of our experiments this was round 5—10 mm² (wire 0.6 mm dia., effective length 6—8 mm on the explosive or 10—20 mm in the air). Away from the electrodes the area is of course larger and may be taken at a rough guess as lying between 0.1—1 cm².

The current density vector, **j**, is given in its general form by

$$\mathbf{j} = \Sigma n_s e_s \bar{\mathbf{c}}_s \qquad (2)$$

where n_s is the number of charge carriers of type s per unit volume, e_s their charge, $\bar{\mathbf{c}}_s$ their average velocity.

Of these factors e_s may be taken as equal to the charge of the electron, since at the temperatures involved the probability of double ionization is quite small. The number of charge carriers, n_s, depends on the state of the gas. If a state of equilibrium is assumed, with equipartition of energy for the various particles such as ions, molecules and electrons, then the number of charge carriers may be calculated by Saha's theory[9].

From Saha's theory we get:

$$\alpha_i^2/(1-\alpha_i^2) = [(kT)^{5/2}/p]\ (2\pi m/h^2)^{3/2}\ e^{-U_i/kT} \qquad (3)$$

where α_i is the ratio of ionized particles to the total number of particles, k is the Boltzmann constant, T is the absolute temperature, p is the pressure, m is the mass of the electron, h is Planck's constant and U_i is the ionization energy. The values of p and T can be calculated from the measured velocities of detonation and of shock waves (see

Taylor[10] and Travers[11]). When the number of charge carriers per cm³ was calculated by this formula for a cylindrical charge of PETN weighing 28 grams for which velocities data were available, a figure round 10^{16} was found for the detonation zone, 10^{15} immediately behind the shock front next to the charge, 10^{13} at a distance of some 8 cm from the charge, and 10^{11} at 15 cm from the charge. Assuming on the other hand that there is no equilibrium, as is asserted by Saenger and others[12,13,14], the number of charge carriers will be larger. According to Saenger's theory the energy from the detonation or from the shock wave is at first transferred to the undisturbed gas as a uniform translational energy. Within the first ten molecular collisions a Maxwellian velocity distribution is established with an average velocity equal to the translational velocity. Those molecules with a kinetic energy equal to or higher than their ionization energy become ionized; their number is given by the expression:

$$n_i/n = \sqrt{6/\pi}\,(c_i/c_t)\,e^{-3\,c_i^2/2\,c_t^2} + (2/\sqrt{\pi}) \int_{\sqrt{3/2}\,(c_i/c_t)}^{\infty} e^{-x^2}\,dx \qquad (4)$$

where n_i is the number of ions per cm³, n is the number of molecules per cm³, c_i is the average velocity corresponding to the energy of ionization and c_t is the average translational velocity. Equilibrium with other energy modes (rotation, vibration, dissociation, electronic excitation and ionization) is gradually reached later. The time taken to reach full equilibrium corresponds to about 10^9 collisions[15] and is therefore of the order of 10^{-2} sec to 10^{-5} sec according to the pressure and temperature. The number of charge carriers in the detonation zone, calculated according to this theory, was of the order of 10^{20} per cm³. In the air the number was 10^{17} next to the charge, 10^{15} at a distance of some 8 cm from the charge and 10^{11} at 15 cm from the charge.

The most difficult quantity to estimate in formula (2) is the factor $\bar{c}_s$, which is the average velocity of the charge carriers. This velocity depends on the mass velocity of the gas, the field strength, the mass of the particles, and the gradients of pressure, temperature and concentration. This velocity may be separated into two components[16], the mean mass velocity of the gas $\mathbf{c}_0$ and the mean velocity of the charge carriers relative to the gas, $\mathbf{C}_s$. At this stage some simplifying assumptions are made; we neglect the effects of the gradients of pressure, temperature and concentration. We may now say that the velocity $\mathbf{C}_s$ is equal to the product of the mobility of the charge carriers K_s and the field strength $\mathbf{E}$.

$$\bar{\mathbf{C}}_s = K_s \mathbf{E} \qquad (5)$$

In our experiments the mobility of the electrons was much higher than that of the ions, so that the contribution of the ions to the current can be neglected, in comparison with that of the electrons. If in addition we neglect the effect of the mass flow, relation (2) for the current density is simplified and may be written as:

$$\mathbf{j} = n_e e K_e \mathbf{E} \qquad (6)$$

where K_e is the mobility of the electrons, n_e the number of electrons per cm³ and e the electron charge. The mobility of the electrons is given by[17]:

$$K_e = 0.921\,\lambda e \Big/ \sqrt{m}\left[kT + [k^2T^2 + (\lambda^2 M X^2 e^2/1.33\ \mathrm{m})]^{1/2}\right]^{1/2} \qquad (7)$$

where m is the mass of the electron, λ is the mean free path of the electron, e is the charge of the electron, k is Boltzmann's constant, T is the absolute temperature, M is the mass of the gas molecule and X is the field strength. When calculated by this formula, the mobility of the electron in the detonation zone is of the order of 10^{-1} cm/sec. In air it increases from 10^2 to 10^3 cm/sec per V/cm when the distance from the charge is increased from 3 to 15 cm.

The number of charge carriers required to produce currents similar to those observed by us experimentally may now be estimated as follows. The conducting cross sectional area was estimated earlier as 0.1—1.0 cm^2. The field strength is obtained by dividing the voltage drop on the electrodes by the distance between them (neglecting the distortion of the field caused by space charge). The mobility of the electrons is calculated by formula (7). The resulting figures, which can be taken only as a rough estimate, are of the order of 10^{20} electrons per cm^3 in the detonation zone of PETN, 10^{17} at a distance of 3 cm from the charge, 10^{15} at 8 cm and 10^{14} at 15 cm from the charge. The figure obtained for the explosive agrees both with the assumption of inequilibrium and with results obtained by Saenger[18]. The figures for the air agree with the above theory except for the last figure which is of a higher value.

It may thus be concluded that the gas in the detonation zone and in the shock wave close to the charge is not in a state of thermal equilibrium.

EXPERIMENTAL

The experimental arrangement for measuring conductivity between two electrodes on an explosive charge is shown schematically in Figure 1.

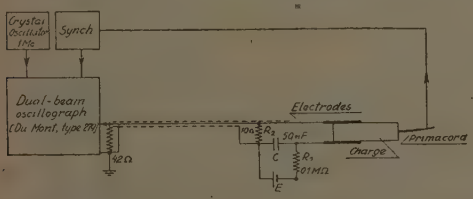

Figure 1
Schematic arrangement for measuring conduction between electrodes on the charge

One electrode is connected directly to one deflection plate of the oscillograph (Dumont type 279, dual-beam) by a shielded cable, which is terminated with its characteristic impedance (42 Ω). The other electrode is connected through a large capacitor (50 microfarads) to the shield of the cable, which is connected to the other deflection plate and grounded. The capacitor is charged to some voltage, E, through a high resistance, R_1. When conduction starts the capacitor discharges through the electrodes and the termination resistance, but its voltage hardly drops during the short period of conduction. When stronger currents through the electrodes are desired, a low resistance, R_2, (10 Ω) may be added to the field end of the cable.

The detonation of the charge was initiated by primacord which also served to provide a synchronizing signal to the oscillograph. The synchronizing signal was obtained from a probe which consisted of two copper wires threaded through the primacord at the right distance ahead of the charge, corresponding to the desired time advance.

Single sweep oscillograms were recorded showing the current fluctuations during the explosion, one beam serving as a time marker with a frequency of 1 MC. A typical

oscillogram is shown in Figure 2. The electrodes, which were copper wires 6 cm long, were laid along a rod of Pentolite in the direction of the advancing detonation front. The current is seen to rise to some constant value in about one microsecond, to stay there for about eight microseconds and to fall back to zero in another two microseconds. The time during which the current remains constant agrees well with the time taken by the detonation front to traverse the electrodes. Experiments which will be described later show that an increase in the area of the electrodes causes the current to increase. It must therefore be concluded that in the present case the conducting area, and hence the conducting length, remains constant while the electrodes are swept by the detonation front. The current rise-time represents the stage when this area is still increasing. The increase in area and length continues for one microsecond, which corresponds to a movement of 7 mm by the detonation front. The fact that the current does not continue to rise after this initial rise-time means that conduction takes place only in a zone about 7 mm deep. Possible explanations are that the electrodes inside the detonation zone remain intact only for about one microsecond before they disintegrate, or that there is a zone of a much higher conductivity, which is about 7 mm deep, immediately behind the detonation front.

Results from a number of similar oscillograms, but with different voltages across the electrodes, are given in Table I. The voltages on the electrodes were calculated from the recorded current and the known battery voltages, taking into account the distributed resistance of the cable (2 Ω).

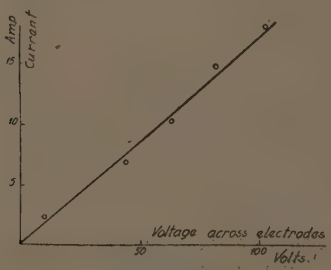

Figure 3
Variation of current with voltage across electrodes on the charge. Copper wire electrodes 0.6 mm diameter, 20 mm apart on Pentolite 50/50

TABLE I

Variation of current with voltage.
0.6 mm dia. copper wire electrodes along the sides of a 20 mm dia. rod of Pentolite 50/50

Voltage on electrodes (volts)	*Current (amperes)*	*Ratio of voltage to current*
10.6	2.4	4.4
43.9	6.9	6.4
62.3	10.3	6.0
80.5	14.7	5.5
102.3	18.1	5.6

The variation of current with voltage is given graphically in Figure 3 and is seen to be practically linear, the ratio of voltage to current being 5.5 within 20 per cent.

This seems to indicate that formula (6) holds in the range investigated. The ratio of voltage to current may conveniently be regarded as "ionization resistance". The conduction during the explosion may thus be visualized as the closing of a switch in the circuit with an appropriate resistance connected across the electrodes (Figure 4). In the previous case this resistance was about 5.5 ohms. When thin electrodes are chosen and are carefully arranged

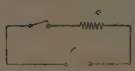

Figure 4
Schematic representation of arrival of detonation at the electrodes

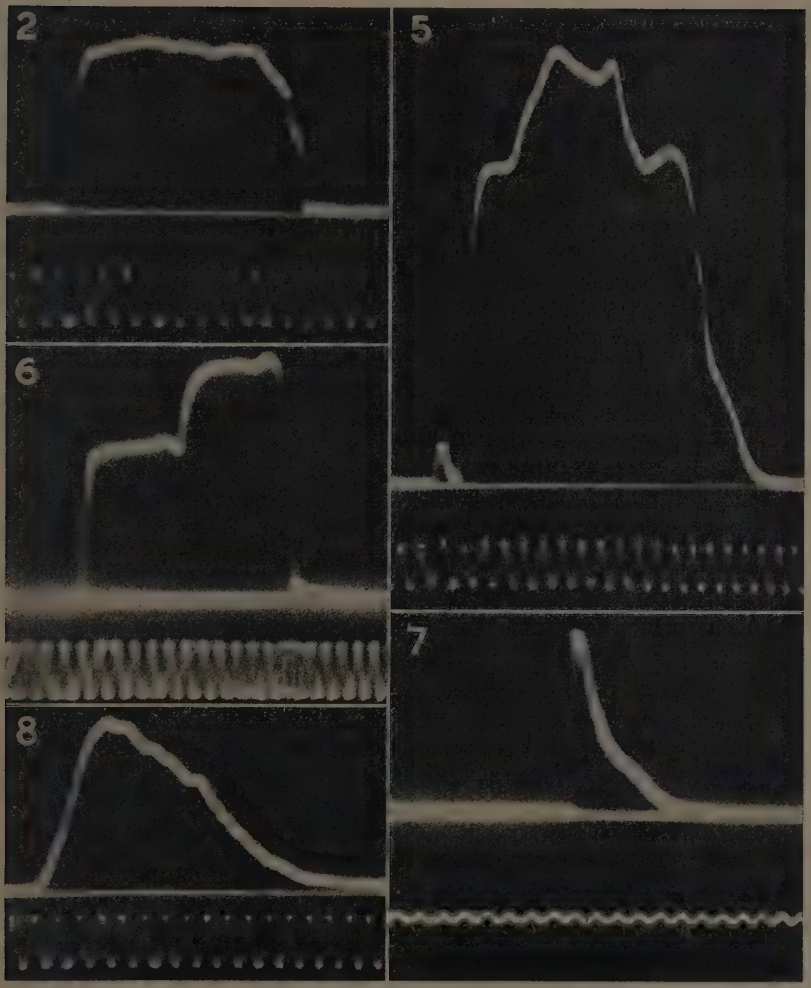

Figure 2; Conduction between electrodes on detonating explosive. Copper wire electrodes 0.6 mm diameter. 60 mm long, along a Pentolite 50/50 rod 20 mm diameter. Time marking 1 MC. Battery voltage 100 V. — Figure 5; Connection in parallel. Conduction of two pairs of copper wire electrodes. 0.6 mm diameter, connected in parallel and placed along two similar Pentolite 50/50 rods, 20 mm diameter, which were detonated one after the other with a delay of 2—3 microseconds .Time marking 1 MC. Battery voltage 200 V. — Figure 6; Effect of area. Conduction between electrodes of varying area, along a Pentolite 50/50 rod 20 mm diameter. First half of the electrodes was copper wire 0.18 mm diameter, the second half was copper foil 11 mm wide. Time marking 1 MC. Battery voltage 100 V. — Figure 7; Electrodes parallel to detonation front. Copper wire electrodes 1.5 mm diameter, 20 mm long, 16 mm apart, inside a charge of PETN. Time marking 1 MC. Battery voltage 100 V. — Figure 8; Conduction in a shock wave in air. Copper wire electrodes 0.6 mm diameter, 16 mm long, 5 mm apart, placed parallel to the shock front, at a distance of 150 mm from a 30 grams charge of PETN. Time marking 1 MC. Battery voltage 250 V.

parallel to the detonation front, it is possible to reduce the current rise-time to a small fraction of a microsecond. Thus the picture of closing a switch is not far wrong.

It was interesting to see whether the ionization resistance behaved like an ordinary resistance and obeyed the rules of addition in series or parallel. To check this, electrodes on two similar charges were connected in parallel, and the charges were fired one after the other within a few microseconds. Figure 5 shows the oscillogram which was obtained in one of the experiments. The shoulders seen on the oscillogram were caused by current flowing separately through each of the probes (14.6 amp). The central portion is the result of the combined current (19.3 amp). The corresponding ionization resistances are 5.4 ohms separately and 2.2 ohms in parallel. Another experiment gave the figures 4.5 and 2.3 ohms respectively. It may thus be seen that the ionization resistance behaves roughly like an ordinary resistance.

The value of the ionization resistance is affected, as might be expected, by the area of the electrodes as well as by the distance between them.

The effect of a change in area is strikingly demonstrated in Figure 6, which shows the change in current accompanying an increase in width of the electrodes halfway down the charge. However, the change in ionization resistance, from 7.3 to 2.2 ohms, was not proportional to the increase in area.

Changes in distance between the electrodes also affected the resistance. For example, a reduction in distance from 20 to 2 millimetres, in 0.6 diameter copper electrodes along a Pentolite rod, caused the ionization resistance to drop from 5.1 ohms to 1.2 ohms.

The observed effects of area and distance on conductivity do not confirm the simple picture presented by formulas (1) and (6), mentioned earlier. It is therefore concluded that the non-homogeneous electric field and the variable conducting area in the ionized medium must play an important part.

The ionization resistance is also dependent on the type of explosive. Thus, values measured on charges of PETN were some 3–4 times larger than those found on Pentolite.

Figure 7 shows an oscillogram of a charge with electrodes parallel to the detonation front. There is a rapid current rise followed by a gradual decay. The rise-time is probably equal to the time in which the detonation front sweeps the electrodes.

When the electrodes are placed near the charge instead of on it, the ionization resulting from the passage of the shock wave in air causes a current to flow between the electrodes. Figure 8 shows such an oscillogram when the electrodes are parallel to the shock front. The general features are similar to those seen in Figure 7, but the times are longer. The rise-time is 2.8 microseconds, which corresponds to a distance of 7 mm travelled by the shock front. Apparently this is the width of the area in which the maximum conduction takes place in this case. Other experimental results supported the idea that the rise-time is equal to the time in which the electrodes and some region round them are swept by the shock wave. The total time of conduction, in this case about 14 microseconds, corresponds to the time in which the electrodes are surrounded by ionized gases.

Results from a number of similar trials but with 1.5 mm between the electrodes and different voltages across them are given in Table II.

The dependence of current on voltage is shown graphically in Figure 9. It is seen that the voltage to current ratio changes only by a factor of about two over the range

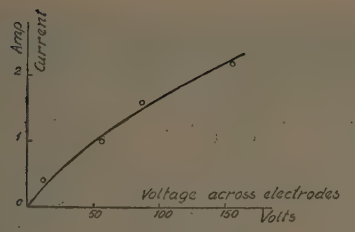

Figure 9

Variation of current with voltage across electrodes in air. Copper wire electrodes 0.6 mm diameter, 16 mm long, 1.5 mm apart, at a distance of 150 mm from a 30 grams charge of PETN

TABLE II

Variation of current with voltage across electrodes in air

Voltage (volts)	*Current (amperes)*	*Ratio of voltage to current*
12	0.4	30
56	1.0	56
87	1.6	54
155	2.2	70

of voltages tried; it therefore seems justifiable to retain the term "ionization resistance" in describing conduction in shock waves.

Results obtained with different distances between charge and probe are given in Table III.

TABLE III

The effect of distance from the charge on conduction. Electrodes 0.6 mm dia., 16 mm long and 5 mm apart at various distances from a charge of PETN weighing 30 grams. Battery voltage 250 volts

Distance from centre of charge (mm)	*Voltage on electrodes (volts)*	*Current (amperes)*	*Ionization resistance (ohms)*	*Current rise-time (microseconds)*
51	26	4.8	5.4	0.4
93	112	3.1	36	0.8
163	191	1.3	144	2.8
216	246	0.09	2700	—

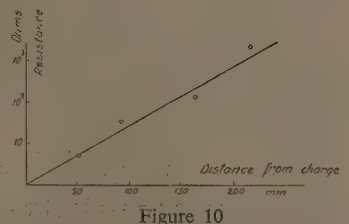

Figure 10

Variation of "ionization resistance" with distance from charge. Copper wire electrodes 0.6 mm diameter, 16 mm long, 1.5 mm apart

When the distance is increased, both current rise-time and ionization resistance increase, the latter increasing roughly exponentially with distance (see Figure 10). When the distance was kept constant the ionization resistance was found to decrease when the weight of the charge was increased. The addition of a reflector immediately behind the electrodes, which increases the peak pressure and temperature in the reflected wave, also caused a drop in the observed resistance. It thus seems clear that the ionization resistance is a function of the peak pressure and temperature in the shock wave.

When the electrodes are in the air, changes in their area or the distance between them have a similar effect on ionization resistance as when they are placed on the explosive. Thus, for example, the trebling of area, by taking three wires in parallel for each electrode instead of one, caused the resistance to drop from 365 ohms to 177 ohms. Some results from experiments with different distances between the electrodes are given in Table IV.

TABLE IV

Variation of conduction with distance between the electrodes. 0.6 dia. copper electrodes, 16 mm long at a distance of 150 mm from a charge of 30 gm PETN. Battery voltage 250 volts

Distance between electrodes (mm)	*Voltage on electrodes (volts)*	*Current (amperes)*	*Ionization resistance (ohms)*	*Current rise-time (microseconds)*
10	223	0.61	366	3.2
5	191	1.33	145	2.8
1.5	145	2.15	72	2.2

The results of the investigation of conduction in probes are summarized below.

1. The ratio of voltage to current between two electrodes in the detonation zone is practically independent of the voltage in the range investigated, and may conveniently be regarded as "ionization resistance". This resistance becomes smaller when the area of the electrodes is increased or the distance between the electrodes is decreased. It is also dependent on the type of explosive used.

2. In air the picture is generally similar to that on the explosive except that the resistance is no longer independent of the voltage but increases slowly with it. The resistance is strongly affected by the peak pressure and temperature in the shock wave and it increases roughly exponentially with the distance from the charge.

3. The current rise-time depends on the speed with which the electrodes are swept by the advancing front. This time interval may be cut down by reducing the size of the electrodes and spacing them closer together, by arranging them parallel to the shock front, and if possible by making the front move faster.

THE MEASUREMENT OF DETONATION AND SHOCK VELOCITIES

The velocity of a detonation front in an explosive or of a strong shock wave in air may be measured by placing probes along the route of the wave at the desired intervals and recording the resulting signals on a suitable oscillograph; the signals are formed in each probe in turn on the arrival of the ionized wave. The method of recording adopted by us consisted of generating a single sweep spiral time base on the oscillograph and marking the arrival of the probe signals on it by blanking the beam for a short time (1/4 microsecond). The use of circular or spiral time bases for the accurate measurement of time intervals is well known[19,2]. A circular time base with a period of 10 microseconds may have an accuracy of 10^{-8}sec[20]. Spiral time bases are advantageous when relatively long time intervals are to be recorded with high accuracy. The spiral generated in our equipment had some ten to fifteen periods of 10 microseconds each, thus a total time interval of over 100 microseconds could be recorded with an accuracy approaching 10^{-8} seconds.

The spiral time base was generated by giving the X and Y inputs of the oscillograph two sinusoidal voltages at a frequency of 100 KC, which were 90 degrees out of phase and were gradually decreasing in amplitude. Figure 12 is an oscillogram of the spiral time base, as is Figure 11, but with 16 blanking pulses per revolution. The blanking pulses were generated by a 1.6 MC crystal oscillator and were used for calibration. The time base is triggered by a signal from a probe on the primacord which sets off the charge. The distance of this probe from the charge is chosen according to the time ad-

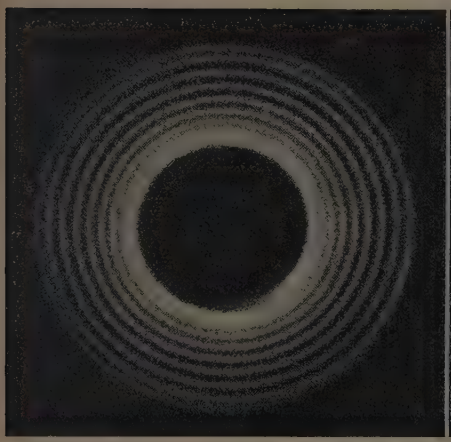

Figure 11
Oscillogram of a spiral time base

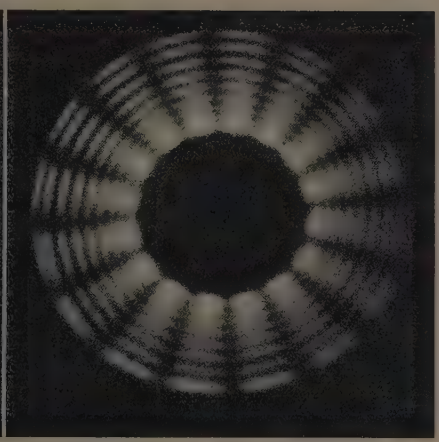

Figure 12
Oscillogram of a spiral time base, with 1.6 MC blanking pulses for calibration

vance required. The amplified signal from the probe fires a thyratron which in turn trips a monostable multivibrator and causes the spiral sweep to appear on the screen.

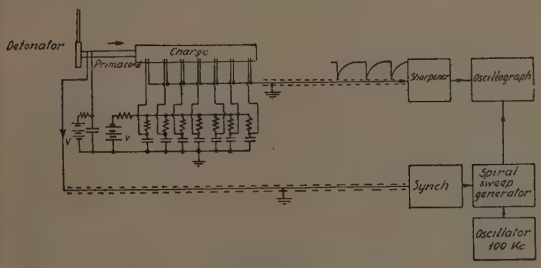

Figure 13
Schematic arrangement for measuring detonation velocity

The arrangement for measuring detonation velocity in solid charges is shown schematically in Figure 13. Two cables are required, one for the synchronizing signal and the other for the blanking signals. Each probe on the explosive is connected to a small capacitor (100 $\mu\mu$ F) which is charged through a high resistance (200 K Ω) to about 60 volts. When the detonation sweeps the electrodes the capacitor discharges through the ionization resistance and the termination resistance in a fraction of a microsecond and the cable is left free to transmit the next signal.

The measurement of velocities of shock waves in air may be carried out in an identical manner. Steps must be taken, however, to amplify and sharpen the signals. The addition of reflectors behind the electrodes is of some help, as is the addition of a sharpener at the oscillograph end. A more radical improvement is obtained, however, by placing a small quantity of a sensitive explosive, such as lead azide, between the electrodes of the probe. When the shock wave is strong enough, this explosive detonates instantaneously on the arrival of the shock front[21] and the signal from the probe is similar to that obtained on the charges. This method worked well up to distances of some 30 cm from a

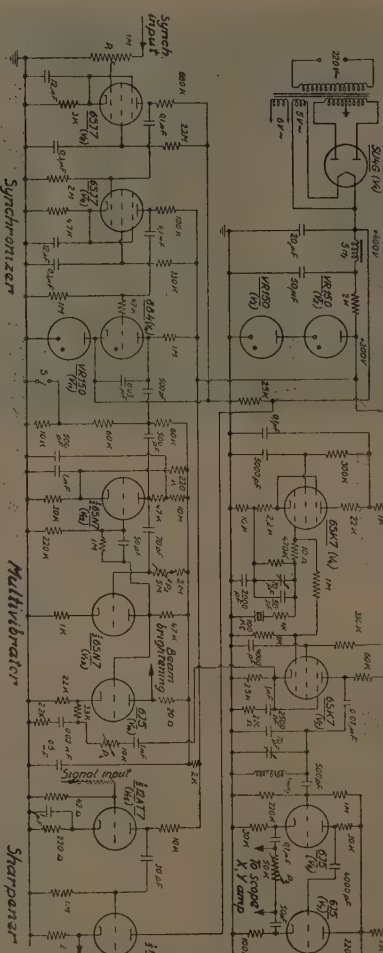

Figure 14

Electronic circuit diagram of spiral time base generator, synchronizer and sharpener

28 gram charge of PETN corresponding to a peak pressure in the shock wave of some 20 atmospheres.

The electronic circuit diagram is given in Figure 14. A regulated voltage of +300 volts and an unregulated voltage of +400 volts are provided by a power supply (V_1, V_2, V_3). The output from a 100 KC crystal controlled oscillator (V_4) is fed into a modulator (V_5) and thence into a phase shifting circuit (V_6, V_7) where two sinusoidal voltages are obtained with a phase difference of 90 degrees. These voltages are applied to the X and Y inputs of the oscillograph to form a circle. A spiral is obtained when the amplitude of these voltages is made to fall off linearly by changing the cathode voltage of the modulator (V_5) with the aid of a multivibrator (V_{12}, V_{13}, V_{14}). When a negative signal is given to the grid of V_{13} a positive square wave is obtained in the anode of V_{13} and a negative square wave in the anode of V_{14}. The negative square wave is used for beam brightening. The positive square wave charges the cathode capacitor of V_5 and modulates the amplitude of the signal from the oscillator, thus causing the spiral to be formed. An RC network in the cathodes of V_{14} and V_5 is used for obtaining a uniform spacing in the spiral; the spacing may be altered with potentiometer P_4. The ellipticity of the spiral is controlled by potentiometer P_2 and by the X and Y amplifiers of the oscillograph. The number of turns (duration) may be varied by potentiometer P_3. In operation, the synchronizing signal from outside, amplified by a two stage amplifier (V_8, V_9), fires a thyratron (V_{10}). The thyratron in turn triggers the multivibrator and the spiral is caused to appear on the screen. The triggering may also be done by closing a switch (S). The signals from the sensing probes are first sharpened by an amplifier (V_{15}) and are then fed via a cathode follower (V_{16}) to the Z input of the oscillograph.

APPLICATIONS

A few examples are given here which illustrate the use of our method in the measurement of detonation and shock wave velocities.

The detonation velocity of primacord was measured with 23 probes spaced 2 cm apart along the cord. Each probe was made of two copper wires, 0.18 mm diameter, threaded through a needle hole at right angles to the primacord and pulled apart to prevent contact between them. The oscillogram which was obtained is shown in Figure 15 and the times measured from it in Table V.

TABLE V

Detonation of primacord.
Time of arrival of detonation at various probes along the primacord

Distance (mm):	20.5	40.2	60.1	100.2	120.2	140.0	160.2
Time (microsec.):	3.25	6.32	9.47	15.78	18.80	22.12	25.29
Distance (mm):	180.0	200.0	220.0	239.9	260.0	280.0	299.9
Time (microsec.):	28.30	31.47	34.63	37.70	40.92	44.16	47.19
Distance (mm):	320.0	340.0	360.0	380.0	400.0	420.6	440.2
Time (microsec.):	50.29	53.60	56.67	59.80	63.13	66.28	69.34

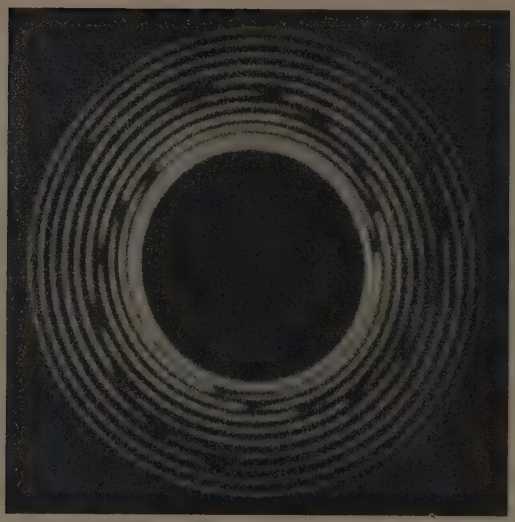

Figure 15
Oscillogram for determining detonation velocity in primacord

As was expected a constant velocity was found (Figure 16); the value of this velocity is 6350 ±10 m/sec. When velocities are calculated between neighbouring probes the results deviate from the average by not more than 3%. This deviation is equal to the experimental error introduced in measuring the distances (±0.3 mm) and the times (±0.05 microseconds).

The detonation velocity in a Pentolite rod, 2 cm diameter and 14 cm long, was measured in a similar way. The probes, 0.18 mm diameter copper wires, were fixed on a thin cardboard base at 1 cm intervals and were placed in contact with the charge. A uniform velocity of 7410 ±40 m/sec was obtained (Figure 17).

The velocity of a shock wave produced by a cylindrical charge of 30 grams PETN

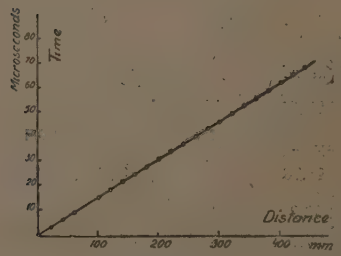

Figure 16
Propagation of detonation in primacord, measured from Figure 15

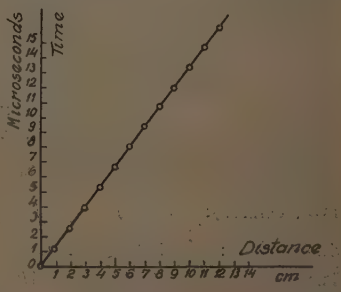

Figure 17
Propagation of detonation in Pentolite 50/50 rod 20 mm diameter

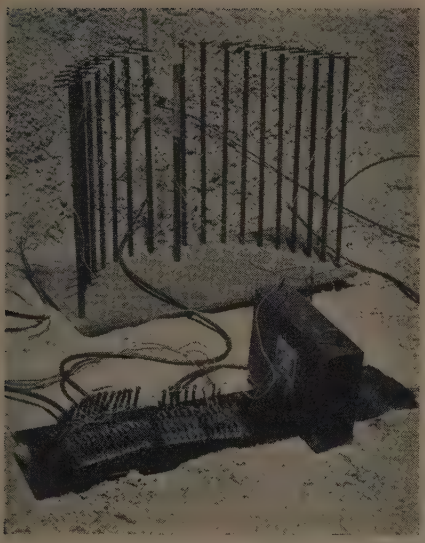

Figure 18
Experimental arrangement for measuring shock wave velocity in air

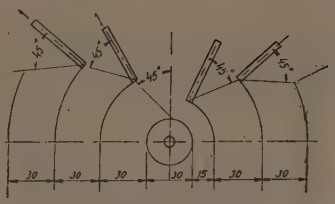

Figure 19
Schematic arrangement of electrodes round charge

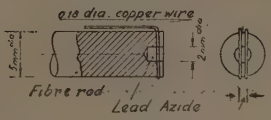

Figure 20
Details of probe for sensing shock waves

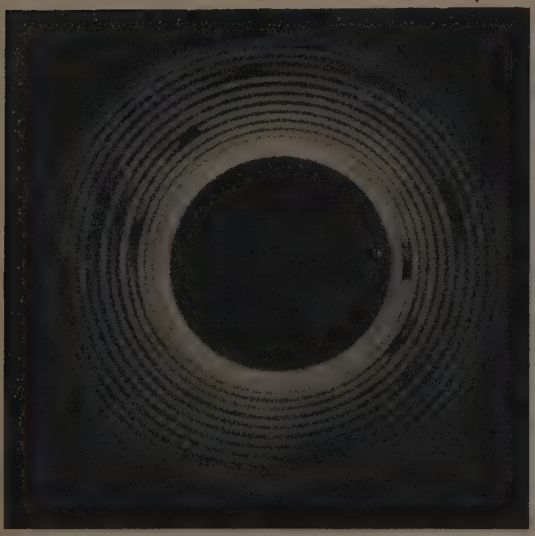

Figure 21
Oscillogram for determining shock wave velocity in air. The shock wave was produced by a 30 grams cylindrical charge of PETN

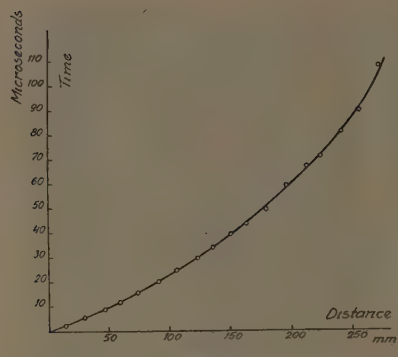

Figure 22

Propagation in air of a shock wave produced by 30 grams cylindrical charge of PETN, measured from Figure 21

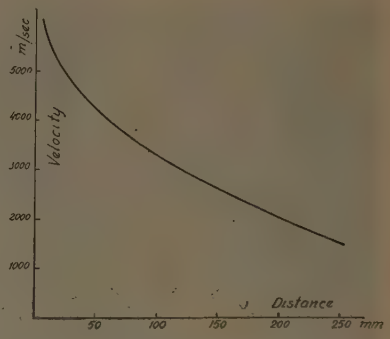

Figure 23

Variation of shock wave velocity in air with distance from a 30 grams cylindrical charge of PETN, calculated from Figure 22

was measured on the plane of axial symmetry with the aid of 19 probes. The arrangement of the probes round the charge is shown in Figure 18. The probes are arranged radially round the charge at distances increasing by steps of 1.5 cm. This particular arrangement, which is seen more clearly in Figure 19, was chosen to ensure that the direct shock wave would reach each probe without interference. This arrangement would also show up any lack of symmetry in the propagation of the shock front, for in that case the experimental results when plotted graphically would fall alternately on two curves instead of one. Details of the probes are shown in Figure 20; it shows two wire electrodes surrounded by lead azide at the end of a fibre rod. Figure 21 shows the oscillogram which was obtained and Figure 22 gives graphically the times that were measured from it. The points on the graph do not alternate about the curve, which shows that the shock front propagated with a radial symmetry. The velocity of the shock wave can be obtained from the slope of this curve (Figure 23), and the peak pressure in the shock wave may be calculated from this information.

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SOME THERMODYNAMIC PROPERTIES OF FREON-22

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The thermodynamic properties of the various Freon compounds have been expressed by the Beattie-Bridgman equation, which, however, does not permit an explicit solution for the specific volume v. The engineer will usually look for a formula representing the volume as function of pressure and temperature, and instruction in thermodynamics will have to dwell on the difference between the ideal and real volumes. The clearest expression of this difference seems to be the form of equation which was used first by Eichelberg for steam[1] and subsequently applied to ammonia[2] as well as again to steam in Mollier's and later tables[3]. All these equations include a series of members with pressure functions as numerator and temperature functions as denominator, derived from an expression for the specific heat c_p, substantially similar to eq. (1a) below.

It seemed interesting to develop such formulae for the Freon-compounds, and Freon-12 was dealt with first. While computations were in progress, information about the preparation of new tables[4] was received; work on Freon-12 was suspended, and the results published here refer to Freon-22 (Monochlorodifluoromethane $CHClF_2$).

On the basis of the well-known tables[5] the specific heat at constant pressure c_p was determined as difference of the enthalpy values and expressed by

$$\frac{c}{p_0} = A + B\frac{T}{100} \tag{1}$$

$$\frac{c}{p} = \frac{c}{p_0} + C\frac{P^2}{(T/100)^3} + D\frac{P^{1,4}}{(T/100)^5} - E\frac{P}{(T/100)^2} \tag{1a}$$

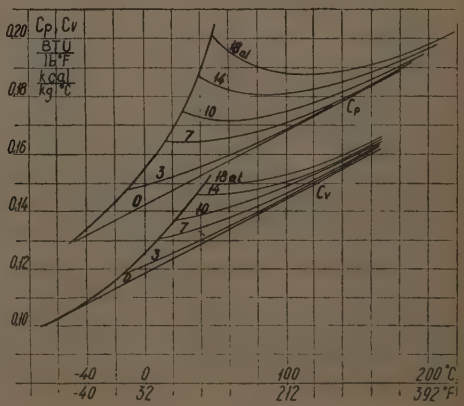

Figure 1
Specific heats of Freon-22 at constant pressure (c_p) and constant volume (c_v)

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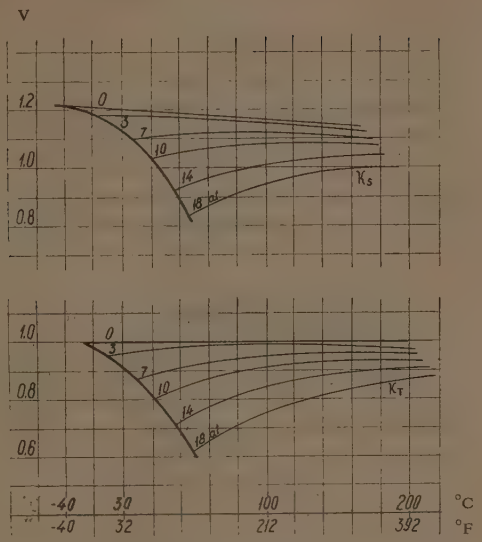

Figure 2

Exponents $\varkappa_T$ and $\varkappa_s$ of isothermal and isentropic changes of state, for Freon-22.

(For numerical values of the constants, see table below). The trend of the c_p isobars, Figure 1, corresponds to the curves known for steam and ammonia but originally (in 1896) determined theoretically by Linde[6] for air. The equation reproduces the table values with a maximum error of about ± 1.5 percent near saturation; at high temperature (300°F), the error increases slightly. It may be noted that the current tables for Freon-12 do not result in the typical decrease of c_p with slight superheat, but curves showing the expected trend have already been published[7].

With the aid of the Clausius equation

$$\left(\frac{\partial c_p}{\partial P}\right)_T = -T\left(\frac{\partial^2 v}{\partial T^2}\right)_P \qquad (2)$$

the equation of state for v can be obtained by differentiation and integration:

$$v = \frac{RT}{P} - \left[C_1 \frac{P}{(T/100)^2} + D_1 \frac{P^{0.4}}{(T/100)^4} - \frac{E_1}{(T/100)}\right] - \left(\frac{F}{P} + G\right) \qquad (3)$$

The auxiliary function $(F/P+G)$ which enters during the integration, was determined in such a manner that the published values of the dry saturated vapour volume were rendered correctly. The error remains far below 1 percent.

On the basis of these equations, it was possible to determine also the exponent $\varkappa_s$ of isentropic changes of state where $Pv^{K_s} = \text{const}$. The usual assumption that this exponent equals the ratio of the specific heats c_p/c_v, holds true for ideal gases only. For real gases, the ratio must be multiplied by a factor $\varkappa_T$ which may be defined as exponent of the isotherm

$$\varkappa_T = -\frac{v}{P\left(\frac{\partial v}{\partial P}\right)_T} \qquad (4)$$

The general equation

$$c_p - c_v = -T \frac{\left(\frac{\partial v}{\partial T}\right)_P^2}{\left(\frac{\partial v}{\partial P}\right)_T} \qquad (5)$$

leads to values for c_v as shown also in Figure 1, and the desired result is

$$\varkappa_s = \varkappa_T \frac{c_p}{c_v} \qquad (6)$$

The explicit form of eq. (4), (5) and (6) is rather bulky and of little interest, but the results as represented in Figure 2 show clearly the expected influence of pressure and temperature. Similar results have been demonstrated for steam[1] and ammonia[8].

TABLE I
Constants for equations (1), (1*a*) *and* (3)

	British	Metric		British	Metric
A	72.4 × 10⁻³	72.4 × 10⁻³	C_1	4.93 × 10⁻³	1.35 × 10⁻⁷
B	14.28 × 10⁻³	25.7 × 10⁻³	D_1	4.75	2.06 × 10⁻³
C	0.027 × 10⁻³	0.95 × 10⁻¹¹	E_1	0.3985	13.8 × 10⁻³
D	12.7 × 10⁻²	69.0 × 10⁻⁸	F	3.54	155.0
E	1.47 × 10⁻³	6.46 × 10⁻⁷	G	56.1 × 10⁻³	3.50 × 10⁻³
			R	17.847/144	9.806

For use of "British" constants, pressures in psia, volumes in cu ft/lb; for "metric" constants, pressures in kg/sq m.

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ON A GENERALIZATION OF UNIFORM STRUCTURES AND SPACES

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I. INTRODUCTION

The following is an abstract study belonging to pure mathematics, and no claim to successful application to physical theory is made. The considerations of the introductory section are only to suggest, eventually, to physicists, some possible lines of interest in conceptual research. At least, they served the author as an heuristic guide for his mathematical speculation.

Generalized concepts of real space play an important role in fundamental research concerning the theory of relativity, but only metric spaces seem to have been considered in this connection. A somewhat more general form of space, still very similar to metric space, is known as *uniform space*[1]. It does away with the exclusively numerical character of the *distance* between two points A and B — even very *near* ones — but still conserves the symmetry with respect to A and B. On the other hand, general as well as physical considerations concerning the irreversibility of time, of the relation of cause and effect, of entropy, the essential importance of irreversible processes — in biology even more than in physics — make it desirable to have space concepts without this symmetry[2].

Can such an asymmetry be achieved without falling into too great and physically meaningless abstractions, without getting something which is a space only by metaphor? The following reasons seem to indicate that this question might have a positive answer:

1. The well known fact that it is possible to define topologies based on order relations only.

2. Our observation that the proof for the existence of a topological *uniform space* associated with a given *uniform structure* does not essentially need the symmetry of the uniform structure.

3. Our observation that relations of equality can often advantageously be replaced by more general *quasi-order* or *substitution* relations without impairing the validity of important chains of deductions. As a consequence stronger and more general theorems will then be obtained[3].

4. Together with such asymmetrical spaces *semi-group* concepts, replacing the usual group concepts, suggest themselves as natural tools of thought for the treatment of the irreversible character of nature. This is the more significant as there exist infinitely many classes of cancellation semi-groups which do not permit of any extension to whole groups[4].

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5. The very strong analogy between certain aspects of the semi-group word problem, characterizing the embeddability of quasi-ordered semi-groups into quasi-ordered groups — and the process of creation and annihilation of short lived pairs of *anti-particles*, schematically described in a *granulated* time composed of distinct moments[4].

6. A more qualitative theory describing something like order relations instead of equations does not exclude extensive use of numerical concepts. Nothing leads closer to the concept of real number than inequalities[5].

This provisory discussion shows already two possible trends of research, one underlining a continuous (or topological) space aspect, the other a discrete operational algebraic (algorithmic) aspect. We do not possess a really general theory of topological semi-groups — more specifically of *quasi-ordoform semi-group spaces* — bridging completely the gap between these two aspects. However, some developments in this direction exist. Elsewhere, we have treated the extension of a topological semi-group to a topological group of quotients (and also of topological rings into topological fields) in some detail[6]. That really new results in this direction—independently of embedding theorems—may not be easily obtained is perhaps indicated by the following typical theorem: A compact cancellation semi-group is necessarily a (compact topological) group[7].

The purely algebraic aspect has been studied by us extensively and in great generality under the title of quasi-ordered semi-groupoids[8]. Here we shall limit ourselves to a more explicit study on the foundations of quasi-ordoform structures and spaces which were previously only briefly mentioned[8]. Our exposition here should be also readable for non-experts and parallels §§ 1—2 of Bourbaki[1]. The deeper and more complicated questions of completion and of compactness paralleling Bourbaki's §§ 3—4 are not treated, although their generalization in our sense seems not hopeless. In particular, it should be interesting to know how far the important result of Uryson about the essentially "real number" character of uniform space (cf. Bourbaki *l.c.* Chap. IX) generalizes to (quasi-) ordoform space.

II. ELEMENTARY CALCULUS OF RELATIONS

We have to introduce some concepts and notations of the theory of binary relations.

Given any abstract set E we form its *square* set $E \times E = E^2 = \{e_1/e_2\}$, $e_1, e_2 \in E$ (e_1, e_2 run independently through E)[9]; in other words, E^2 means the set of all ordered pairs of elements of E. A subset of E^2 is called a (*binary*) *relation in* (or *over*) E. If $e_1/e_2 \in R \subset E^2$ we say also "e_1 is in the relation R to e_2", and we write $e_1 R e_2$.

As particular relations we mention 1) the *void* relation $\emptyset$ (there is no pair satisfying this relation); 2) the *universal* relation $U = U_E = E^2$ (consisting of all pairs); 3) the *diagonal* relation $\Delta = \Delta_E$ (consisting of all pairs e/e, i.e. with $e_1 = e_2$) also called the *identity* relation in E.

Any two relations in E, say R and S, may be composed or *multiplied* to give a uniquely determined third relation T, written $RS = T$, by the following definition of T: If and only if $e_1 R e_2$ and $e_2 S e_3$ then $e_1 T e_3$; in other words, $a/c \in T$ is equivalent to the existence of a $b \in E$, such that $a/b \in R$ and $b/c \in S$. This composition is evidently not commutative but easily proved to be associative. Multiple products may therefore be written without brackets, and the whole set of relations in E, i.e. the set $\mathfrak{P}(E)$ of subsets (parts) of E^2,

forms a semi-group with respect to this operation, but not a cancellation semi-group. In particular we may speak of the nth power of a relation, written $\overset{n}{R}$, i.e. $R = \overset{1}{R}$, $RR = \overset{2}{R}$, etc. For all R we have:

(1) $R\,\varnothing = \varnothing\,R = \varnothing$ (the product of two non-void relations may be void too);

(2) $R\Delta = \Delta\,R = R$

$R \subset S$ ("R is contained in S as a proper or improper subset"), is equivalent to "the relation R implies the relation S", i.e. $aRb \longrightarrow aSb$.

Composition of relations is monotonous: $R \subset S \rightarrow RT \subset ST$ and $TR \subset TS$ for all T. As a consequence we have also: $R \subset S \rightarrow \overset{2}{R} \subset \overset{2}{S}$ (Proof: $\overset{2}{R} \subset RS \subset \overset{2}{S}$), and also $\overset{n}{R} \subset \overset{n}{S}$.

A relation R containing Δ, i.e. $R \supset \Delta$, is called a *reflexive* relation. If in a product RS or SR one factor, say R, is reflexive, then $RS \supset S$ and $SR \supset S$ (because of $RS \supset \Delta S = S$, etc.). In particular, the product of two (or more) reflexive relations is again a reflexive relation, i.e. the reflexive relations form a subsemi-group. For reflexive relations we have

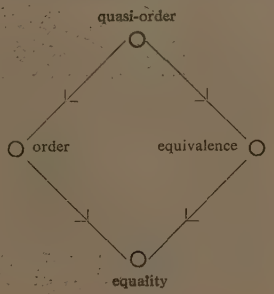
$$\Delta \subset R \subset \overset{2}{R} \subset \overset{3}{R} \subset \ldots$$

If $\overset{2}{R} \subset R$, R is called a *transitive* relation. This is equivalent to the well known defining formula $a\,R\,b,\ b\,R\,c \longrightarrow a\,R\,c$. For transitive relations $R \supset \overset{2}{R} \supset \overset{3}{R} \ldots$ A reflexive and transitive relation Q is called a *quasi-order* and will be written $\longrightarrow$ or $\longleftarrow$. Evidently $Q \subset \overset{2}{Q}$ and $Q \supset \overset{2}{Q}$ implies $Q = \overset{2}{Q} = \overset{3}{Q} = \ldots$

To every given R in E there exists its *inverse* relation, written $\overset{-1}{R}$, obtained from R by inversing the order of every pair in R; $a/b \in R \longleftrightarrow b/a \in \overset{-1}{R}$. $\left(\overset{-1}{R}\right) = R$. Clearly, $\overset{-1}{R}\,\overset{-1}{R} = \left(\overset{-1}{R}\right) = \left(\overset{2}{\overset{-1}{R}}\right)$ and therefore we denote it by $\overset{-2}{R}$; similarly we have $\overset{-n}{R}$. $\Delta = \overset{-1}{\Delta}$ and the inverse of a reflexive relation is again a reflexive relation; further $\overset{-1}{(RS)} = \overset{-1}{S}\,\overset{-1}{R}$ and $R \subset S \longrightarrow \overset{-1}{R} \subset \overset{-1}{S}$. Therefore the inverse of a transitive relation is again a transitive relation. In particular $R \subset \overset{-1}{R} \longrightarrow R = \overset{-1}{R}$, and such an R is called a *symmetrical* relation. A symmetrical quasi-order is called an *equivalence*.

Given any relation R, $R \cap \overset{-1}{R}$ is a symmetrical relation. A quasi-order Q with $Q \cap \overset{-1}{Q} = \Delta$ is called an *order* relation (in the usual sense) and written $\leqq$ or $\geqq$ (in this case a/b, $b/a \in Q \longrightarrow a/b \in \Delta \longrightarrow a = b$); $Q - \Delta$ is called a *strict* order and denoted by $<$ or $>$; if $Q \cup \overset{-1}{Q} \supset E - \Delta$ the order Q is called a *total* order (strict or non-strict).

We have the following lattice (Hasse diagram) showing the order of generality of these concepts.



For any relation $R = \{x/y\}_{x/y \in R}$ and any element $e \in E$ we can form the two so called *cuts*, the *first* or *vertical* one $R_1(e) = \{y\}_{e/y \in R}$ being the set of all $y \in E$ such that $e/y \in R$, and the *second* or *horizontal* cut $R_2(e) = \{x\}_{x/e \in R}$. E.g.: $R_1(e) = \overset{-1}{R}_2(e)$, $R_2(e) = \overset{-1}{R}_1(e)$.

For symmetrical relations the two cuts coincide.

Relations with vertical cuts containing at most one element are called *uniform* or *univalent functions* defined for those *arguments* for which the vertical cuts consist precisely of one element.

The following proposition indicates why the quasi-order should be considered for many purposes as the best possible generalization of the relation of equality.

Let the abstract set E be also a *multiplicative* system, i.e. be provided with a multiplicatively written operation. We may then define several kinds of *compatibility* of relations R over E with the multiplication in E.

The two simplest ones concerning a single relation are:

(1) *Homogeneity* (H) meaning simultaneous *right* and *left* homogeneity $(H) \longleftrightarrow (H_r)$ & (H_l), where

(H_r) means that $aRb \longrightarrow axRbx$ }
and (H_l) means that $aRb \longrightarrow xaRxb$ } for every $x \in E$

(2) *Regularity* (r) means that $a\,R\,b$ & $a'\,R\,b' \longrightarrow aa'\,R\,bb'$.

PROPOSITION: *For a quasi-order homogeneity and regularity coincide.*

Proof: We shall show that a) if R is transitive then $(H) \longrightarrow (r)$ and
b) if R is reflexive then $(r) \longrightarrow (H)$.

a) $a\,R\,b \longrightarrow aa'\,R\,ba'$ (by H_r),
$a'\,R\,b' \longrightarrow ba'\,R\,bb'$ (by H_l), therefore
$a\,R\,b$ & $a'\,R\,b' \longrightarrow aa'\,R\,ba'$ & $ba'\,R\,bb' \longrightarrow aa'\,R\,bb'$ (by transitivity),
i.e. R is regular.

b) $a\,R\,b$, $x\,R\,x \longrightarrow ax\,R\,bx$, $xa\,R\,xb$, i.e. R is (H).

The property (H) or (r) if applied to a quasi-order is also called *monotony*; and we call a *monotonous quasi-order* also a *substitution* relation. E.g.: The relation of set theory $\subset$ ("is contained in", or "contains") is a substitution relation in the multiplicative system of relations.

III. QUASI-ORDOFORM BASES

Let Φ be a non-empty (generally infinite) family of reflexive relations over a set E. Φ will be called a *quasi-ordoform base over E* if the following two axioms are satisfied:

(1) The so-called Moore-Smith or *filter base* property

$$R, S \in \Phi \longrightarrow (\exists\, T \in \Phi)(T \subset R \cap S).$$

This is the characteristic property of so-called *directed sets*; in other words, we may say that with any two given members R and S of the family there exists a member T in the family smaller than both.

(2) *Collective transitivity*

$$R \in \Phi \longrightarrow (\exists\, R' \in \Phi)(R'R' \subset R),$$

i.e. with every member $R \in \Phi$ there can be found a member $R' \in \Phi$ such that $R'^2 \subset R$.

To understand how these family properties generalize those of individual relations we consider the degenerate case in which the family Φ consists of one single reflexive

relation R, $\Phi = \{R\}$. Then (1) is trivially satisfied because $R \subset R \cap R$; and, by (2), $R' = R$ must be such that $\overset{2}{R} \subset R$, i.e. R is transitive; R must therefore be a quasi-order and every quasi-order is a degenerated quasi-ordoform base.

On the other hand, a *base of a uniform structure*, or shorter a *uniform base*[10], is characterized by the same axioms (1) and (2) and the following additional axiom (3) expressing a property of symmetry.

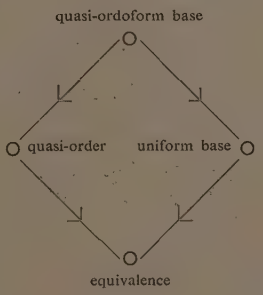
$$(3) \qquad R \in \Phi \longrightarrow (\exists R' \in \Phi)(R' \subset \overset{-1}{R}).$$

If a uniform base is reduced to one member R only, then R must be an equivalence. We may therefore say that the notion of quasi-ordoform base generalizes that of uniform base just as the quasi-order generalizes the equivalence, and with equal right that the quasi-ordoform base generalizes the individual quasi-order relation in exactly the same way as the uniform base generalizes the individual equivalence relation.

Let us write $D_\Phi = \bigcap_{R \in \Phi} R$, i.e. D_Φ is the common part of all members of Φ, where Φ is a quasi-ordoform base. Clearly, $D_\Phi \supset \Delta_E = \Delta$, i.e. D_Φ is itself a reflexive relation, We have even the

PROPOSITION: D_Φ is a *quasi-order*.

Proof: It is sufficient to show that D_Φ is transitive, i.e. that $\overset{2}{D}_\Phi \subset D_\Phi$. Indeed whatever $R \in \Phi$, there exists $R' \in \Phi$ such that $\overset{2}{R'} \subset R$ and therefore, because of $D_\Phi \subset R'$, also $\overset{2}{D}_\Phi \subset \overset{2}{R'} \subset R$ for every $R \in \Phi$, i.e. $\overset{2}{D}_\Phi \subset D_\Phi$ (and therefore even $\overset{2}{D}_\Phi = D_\Phi$).



On the other hand, if Φ were a uniform base, D_Φ would be an equivalence relation. This justifies completely what we said before about the quasi-ordoform base as a generalization of the individual quasi-order as well as of the uniform base concept. We have, therefore the following Hasse diagram ordered as to the degree of generality of the corresponding concepts:

IV. EQUIVALENCE OF BASES: QUASI-ORDOFORM STRUCTURES

This paragraph contains some matters and methods which apply more generally to families of subsets of arbitrary sets as is well known. Certain parts of the discussion are an exact parallel of what could be said by following Bourbaki (*l.c.*) on uniform structures and bases and replacing systematically the word *uniform* by *quasi-ordoform*.

A superfamily $\mathfrak{F}$ of families Φ of subsets of any set F (i.e. $\mathfrak{F} = \{\Phi\}_{\Phi \in \mathfrak{F}}$ $\Phi = \{R\}_{R \in \Phi}$, $R \subset F$; in our case F is a general "square" set E^2), may be organized into a quasi-ordered system by comparing its members Φ in the following well known way: We shall say that "Φ is *finer* than Ψ", $\Phi \prec \Psi$, or equally well "Ψ is *less fine* than Φ", $\Psi \succ \Phi$, if and only if for any member $S \in \Psi$ there can be found a member $R \in \Phi$ such that $R \subset S$; i.e. $\Phi \prec \Psi \longleftrightarrow \Psi \succ \Phi \longleftrightarrow (S \in \Psi)(\exists R \in \Phi)(R \subset S)$.

One has to be careful about the order of the symbols in this definition. The corresponding dual concept of *coarser*, respectively *less coarse*, is *not* equivalent to *less fine*, respectively *finer*, and if simultaneously needed a new symbol, distinct from $\prec$ and $\succ$, has to be invented.

The relation $\succ$ (or $\prec$) is clearly reflexive and transitive and therefore a quasi-order. It is evidently not symmetrical and therefore not an equivalence; on the other hand, there exist distinct families Φ and Ψ, $\Phi \neq \Psi$, such that simultaneously $\Phi \prec \Psi$ and $\Psi \prec \Phi$, i.e. $\prec$ is not an order. But in a well known way[11] every quasi-order defines a canonically associated equivalence $\mathcal{E}$, denoted $\Phi \sim \Psi$, by the definition

$$\Phi \sim \Psi \longleftrightarrow (\Phi \prec \Psi \;\&\; \Psi \succ \Phi).$$

Such an equivalence class of quasi-ordoform bases is called a *quasi-ordoform structure* $\Sigma = \Sigma(\Phi)$ uniquely determined by an arbitrary member representing its class. It is a well known and easily verified theorem[11] that the so defined set of equivalence classes, and therefore also every subset of it, will form an ordered system in the natural way induced by the quasi-order of the Φ's. This set $\mathfrak{S} = \{\Sigma\}_{\Sigma \in \mathfrak{S}}$ of equivalence classes or structures is also called the *quotient set* of $\mathfrak{F}$ modulo $\mathcal{E}$ and denoted $\mathfrak{F}/\mathcal{E} = \mathfrak{S}$. Of course, this order is a partial one, and we have to distinguish between *comparable* and *incomparable* structures.

For the sake of completeness and the convenience of the reader we mention also the analogy of the usual, but less general, method which works with the totality of possible families Φ, i.e. in our case with all possible quasi-ordoform bases. It is then obvious that *with two equivalent families* $\Phi \sim \Psi$ *also their union* $\Phi \sim \Psi$ *belongs to the same class of equivalence* modulo $\mathcal{E}$, *and that, even more, with any finite or infinite set of families of such an equivalence class, also their union belongs to it.*

A quasi-ordoform structure, considered as the set of all its bases, forms itself a (partial) ordered system by the usual inclusion relations between its members (the distinct bases). Therefore every quasi-ordoform structure contains a uniquely determined greatest base Φ which is the union of all families of its class and which may be characterized by the following property:

"With any relation $R \in \Phi$, Φ contains also every relation over E containing R", i.e.

$$(0) \qquad R \subset S,\ R \in \overline{\Phi} \longrightarrow S \in \overline{\Phi}.$$

We may consider the uniquely determined $\Phi = \bigcup_{\Phi \in \Sigma} \overline{\Phi} \in \Sigma(\Phi) = \Sigma(\overline{\Phi})$ as the *normal* or *standard* representantive of Σ and therefore identify Φ with its $\Sigma = \Sigma(\Phi)$.
(E.g.: By this definition $\Phi = \Sigma$ always contains the universal relation $U = U_E$ as an element).

In doing this we could also have defined a quasi-ordoform structure immediately by adding (0) as an axiom. (0) and (1) together constitute the so called *filter* property, as distinct from the *filter base* property (1) only. Clearly we have then as a consequence the stronger and simpler property (1′) (instead of (1)).

$$(1') \qquad R, S \in \overline{\Phi} \longrightarrow R \cap S \in \overline{\Phi}.$$

We could as well say that with any *finite* number of relations $R_1, R_2, \ldots, R_n \in \Phi$ also their intersection $R_1 \cap R_2 \cap \ldots \cap R_n \in \Phi$. The axiom (2) remains without change.

In the particular case of a uniform structure the symmetry axiom for bases (3) will now imply the stronger and simpler property

$$(3') \qquad R \in \bar{\Phi} \longrightarrow \overset{-1}{R} \in \bar{\Phi}.$$

However this second method is not always suitable or may sometimes need special adaptation. Thus, it will often happen that we have to consider only such relations E which are compatible with a certain partition of E^2, i.e. with a certain equivalence $\mathcal{H}$ in E^2; the admissible relations over E will then be set unions of members of the quotient set $E^2/\mathcal{H}$. Secondly, as will be seen in the next paragraph, the method is unsuitable for the definition of the class of structures which we shall call *ordoform structures*. As is evident by considering the italicized statement on the preceding page, these difficulties may be overcome and the usefulness of the method preserved if the superfamily $\mathfrak{F} = \{\Phi\}_{\Phi \in \mathfrak{F}}$ satisfies certain conditions, e.g.:

$\mathfrak{F}$ is closed under the operation of general (i.e. also infinite) set union of its members, i.e.

$$\mathfrak{F}' \subset \mathfrak{F} \longrightarrow \bigcup_{\Phi' \in \mathfrak{F}'} \Phi' \in \mathfrak{F} \quad .$$

Then, in particular, the operation of set union, applied to equivalent bases, will always furnish an equivalent base.

Proof: It is sufficient to prove that $\bigcup_{\Phi \in \Sigma(\Phi_1) \subset \mathfrak{F}} \Phi \sim \Phi_1$. Put $\bigcup_{\Phi \in \Sigma(\Phi_1)} \Phi_1 = \bar{\Phi}_1$.

Clearly $\Phi_1 \subset \bar{\Phi}_1$ and therefore $\bar{\Phi}_1 \prec \Phi_1$.
But one has also $(R \in \bar{\Phi}_1)(\exists\, \Phi \in \Sigma(\Phi_1))(R \in \Phi)$
and $(R \in \Phi \in \Sigma(\Phi_1))(\exists\, R_1 \in \Phi_1)(R_1 \subset R)$.
Therefore $(R \in \bar{\Phi}_1)(\exists\, R_1 \in \Phi_1)(R_1 \subset R)$ i.e. $\Phi_1 \prec \bar{\Phi}_1$,
and finally $\Phi_1 \sim \bar{\Phi}_1$.

Of course, also a weaker condition will do as long as the existence of a greatest family is guaranteed in every equivalence class of families.

Let us put $\mathfrak{F}_{max}/\mathcal{E} = \mathfrak{S}_E$, i.e. let $\mathfrak{S}_E$ be the set of *all* quasi-ordoform structures over E. $\mathfrak{S}_E$ is not only an ordered set, but even a *complete lattice*: every finite or infinite subset $\mathfrak{S}' \subset \mathfrak{S}_E$ has an *upper* and a *lower limit* in $\mathfrak{S}_E$, i.e. a *least fine upper bound* and a *finest lower bound* in $\mathfrak{S}_E$. (What should be called "upper" and what "lower" depends only on arbitrary convention). More explicitly: a) there exists a $\Sigma_1 = \mathfrak{S}' \in \mathfrak{S}_E$ which is finer than every $\Sigma' \in \mathfrak{S}'$, but less fine than any other element of $\mathfrak{S}_E$ with the same property (in other words, Σ_1 is the least fine of all quasi-ordoform structures finer than those of $\mathfrak{S}'$) ; similarly b) there exists $\Sigma_2 = \underline{\mathfrak{S}}' \in \mathfrak{S}_E$, the finest of all $\Sigma \in \mathfrak{S}_E$ which are less fine than those of $\mathfrak{S}'$.

Proof: It is sufficient to prove a) because the lower limit of $\mathfrak{S}'$ is the upper limit of another subset of $\mathfrak{S}_E$, namely that of all lower bounds of $\mathfrak{S}'$. The proof of a) is exactly the same as for uniform structures[12].

V. SEPARATE, DISCRETE AND ORDOFORM STRUCTURES

One verifies easily that

$$\Phi \sim \Psi \longrightarrow D_\Phi = D_\Psi,$$

i.e. that to every quasi-ordoform structure $\Sigma = \Sigma(\Phi)$ there belongs a uniquely determined $D = D_\Phi = D_{\Sigma(\Phi)}$. This justifies the following definition:

Σ is called a *separate* (or *Hausdorff*) quasi-ordoform structure if its $D_\Sigma = \Delta$.

It is a direct consequence of this definition that a separate quasi-ordoform structure generalizes the individual equality relation exactly in the same way as the general quasi-ordoform structure generalizes the quasi-order and the general uniform structure the equivalence. Of course the degenerated separate quasi-ordoform structure, reduced to one single relation, must be $\{\Delta\}$.

D is, in general, the intersection of an *infinity* of sets, and, therefore, it does not fall under the influence of the axiom (1) or (1′). It may therefore be or not be in $\bar{\Phi}$. Indeed the interesting case is that in which never $D \in \Phi$. This is exhibited by the following two propositions:

1. $$\Phi \sim \Psi \longrightarrow (D \in \Phi \longrightarrow D \in \Psi),$$

and therefore also $D \in \Phi \longleftrightarrow D \in \Psi$;

i.e. the property of D to belong (or *not* to belong) to a member of an equivalence class of bases is a class property common to all its members, or, in other words, it is a property of the quasi-ordoform structure itself. This justifies the following *definitions*: $\bar{\Phi}$ is called a *trivial* quasi-ordoform structure if $D_\Phi \in \Phi$, i.e. $\bar{\Phi}$ is a trivial quasi-ordoform structure if and only if it contains a *smallest* relation (common to all bases of Φ).

$\bar{\Phi}$ is called *discrete* if it is separate and trivial.

2. $$D_\Phi \in \Phi \longrightarrow \Phi \sim \{D_\Phi\}.$$

Therefore, if $\bar{\Phi}$ is a trivial quasi-ordoform structure, then, it possesses also a base consisting of one single relation only, which is of course a quasi-order, and we are, in fact, again in the trivial degenerated case of structures reduced to one individual relation.

VI. THE ORDOFORM STRUCTURES

Contemplating the quadrangle Hasse diagram (I) of individual relations and the fact that the general quasi-ordoform structure, the general uniform structure and the separate structures correspond respectively to three of the four verices of (I) we are led to ask if it is possible to intercalate between the general quasi-ordoform and the separate quasi-ordoform structures another class of structures Ω with bases Ω which we will call *general ordoform structures*, respectively *general ordoform bases*, generalizing the order relation exactly in the same way as the general uniform structure generalizes the equivalence relation. For this the only condition required is that $D_\Omega = \bigcap_{R \in \Omega} R$ be an order, i.e. that $D_\Omega \cap D_\Omega^{-1} = \Delta$ (of course $D_\Omega \cap D_\Omega^{-1} \subset \Delta$ is sufficient). As for any indices set I $(\bigcap_{i \in I} R_i)^{-1} = \bigcap_{i \in I} R_i^{-1}$, we have $D_\Omega \cap D_\Omega^{-1} = (\bigcap_{R \in \Omega} R) \cap (\bigcap_{R \in \Omega} R^{-1}) = \bigcap_{R \in \Omega} (R \cap R^{-1})$. Let us write, for any relation R, $R \cap R^{-1} = \sigma_R$ and call σ_R the symmetrical *part* of

R. Our condition will therefore be that $\bigcap_{R \in \Omega} \sigma_R = \Delta$, i.e. that the common part of the symmetrical parts be the identity relation.

We can give another form to this condition as a consequence of the

PROPOSITION: *The symmetrical parts of the relations of any quasi-ordoform base* Φ *constitute a uniform base.*

Proof: Put $\{\sigma_R\}_{R \in \Phi} = \Sigma_\Phi = \Sigma$.

(0) All σ_R being reflexive, they constitute a non-empty family of reflexive relations.

(1) Σ has the filter base property because $T \subset R \cap S$ (in Φ) $\longrightarrow \sigma_T \subset \sigma_R \cap \sigma_S$ (in Σ): $\sigma_T = T \cap \overset{-1}{T} \subset (R \cap S) \cap (\overset{-1}{R} \cap \overset{-1}{S}) = (R \cap \overset{-1}{R}) \cap (S \cap \overset{-1}{S}) = \sigma_R \cap \sigma_S$.

(2) Σ has the collective transitivity because $\overset{2}{S} \subset R \longrightarrow \overset{2}{\sigma_S} \subset \sigma_R$:
$\overset{2}{\sigma_S} = \sigma_S \sigma_S = (S \cap S)(S \cap S) \subset SS \cap SS \cap SS \cap SS \subset S \cap S \subset R \cap R = \sigma_R$.

(3) Σ has the symmetry property because $S \subset \overset{-1}{R} \longleftrightarrow \overset{-1}{S} \subset R \longrightarrow \sigma_S \subset \overset{-1}{\sigma_R}$:
$\sigma_S = S \cap \overset{-1}{S} \subset R \cap \overset{-1}{R} = \overset{-1}{\sigma_R} (= \sigma_R)$.

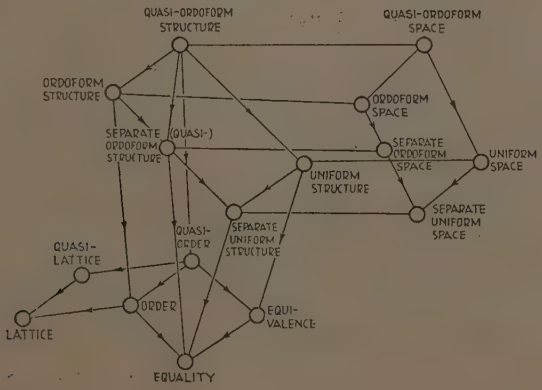
We can therefore reformulate our condition and define:

A quasi-ordoform base $\Omega = \{R\}_{R \in \Omega}$ shall be called a *general ordoform base*, and its corresponding structure a *general ordoform structure*, if and only if the uniform base, constituted by the symmetrical parts σ_R of its relations R, is separate.

In particular, it follows that *for separate structures there is no difference between ordoform and quasi-ordoform*, i.e. a separate quasi-ordoform structure is *ipso facto* a separate ordoform structure. This contains as a special case the separate uniform structures.

Let us note the following important difference between uniform and quasi-ordoform structures: it is not possible to associate canonically to every non-separate (quasi-) ordoform structure a separate one in such a simple way as this is possible for uniform ones. The reason is that D_Φ is not an equivalence.

We may resume our results in the following Hasse diagram giving a complete scheme of the hierarchy of our concepts:



VII. QUASI-ORDOFORM SPACES

A given quasi-ordoform base Φ or structure Σ over E organizes E in two similar but distinct ways as a topological space $\mathfrak{T}_1$ or $\mathfrak{T}_2$, if we interpret for every "point" $e \in E$ either the totality of *first* (left or vertical) cuts or the totality of *second* (right or horizontal) cuts of all the relations $R \in \Phi$, respectively $R \in \Phi \in \Sigma$, as *fundamental systems* or *bases of neighbourhoods*, respectively of *complete systems of neighbourhoods* of the point e, written $\mathfrak{V}_1(e) = \{R_1(e)\}_{R \in \Phi}$ and $\mathfrak{V}_2(e) = \{R_2(e)\}_{R \in \Phi}$, etc.; i.e. every $R_1(e)$ will be considered as a neighbourhood of e in the topology $\mathfrak{T}_1$, and every $R_2(e)$ a neighbourhood of e in $\mathfrak{T}_2$. $\mathfrak{T}_1 = \mathfrak{T}_1(\Sigma)$, respectively $\mathfrak{T}_2 = \mathfrak{T}_2(\Sigma)$ will be called the *first* (*right* or *vertical*), respectively the *second* (*left* or *horizontal*) *topology* deduced from the quasi-ordoform structure Σ, or more briefly the *deduced first*, respectively *second*, *quasi-ordoform space*. Of course, if the quasi-ordoform structures in question are uniform ones, the distinction of the two deduced topologies will disappear as a consequence of the symmetry axiom.

To justify the above ideas we have only to prove that the deduced families of cuts will really be such that they satisfy the axioms for fundamental systems of neighbourhoods. It is of course sufficient to prove this, say, for the first cuts. For this we can follow exactly Bourbaki[12], being only careful about the distinction of the two kinds of cuts.

Proof:

(0) As a consequence of the reflexivity we have $e \in R_1(e)$ for every $e \in E$ and every $R \in \Phi$.

(1) As a consequence of the filter base property we have
$R_1(e), S_1(e) \in \mathfrak{V}_1(e) \longrightarrow (\exists T_1(e) \in \mathfrak{V}_1(e))\,(T_1(e) \subset R_1(e) \cap S_1(e))$.

(2) As a consequence of the collective transitivity we have
$R_1(e) \in \mathfrak{V}_1(e) \longrightarrow (\exists S_1(e) \in \mathfrak{V}_1(e))\,(f \in S_1(e)\ \&\ g \in S_1(f) \longrightarrow g \in \left(\overset{2}{S}\right)_1(e) \subset R_1(e))$.

But these are the three axioms of fundamental systems of neighbourhoods saying:

(0) Every neighbourhood $V(e)$ of any point e contains the point e.

(1) Given any two neighbourhoods of a point there is a neighbourhood of that point contained in the intersection of the given ones.

(2) Every neighbourhood V of a point contains a neighbourhood W of that point such that V is also a neighbourhood of every point of W, or, in other words, such that W is in the "interior" of V.

The trouble with the two, generally distinct, topologies deduced from the same quasi-ordoform structure is less than it might seem at first. It will indeed be sufficient to consider one kind of topology only, say the first one, because to every second topology $\mathfrak{T}_2(\Sigma)$ deduced from a quasi-ordoform structure Σ we can find another quasi-ordoform structure, say Σ^*, such that $\mathfrak{T}_2(\Sigma) = \mathfrak{T}_1(\Sigma^*)$. For this purpose it is sufficient to replace all relations R appearing in the definition of Σ (e.g.: constituting a base Φ of Σ) by their inverses $\overset{-1}{R}$. The second cuts of the R's will be the first cuts of the $\overset{-1}{R}$'s, and therefore $\Phi^* = \{\overset{-1}{R}\}_{R \in \Phi}$ will be a base of a quasi-ordoform structure Σ^* such that $\mathfrak{T}_2(\Sigma) = \mathfrak{T}_1(\Sigma^*)$. It will also be convenient to denote $\Phi^* = \overset{-1}{\Phi}$ and $\Sigma^* = \overset{-1}{\Sigma}$.

VIII. QUASI-ORDOFORM CONTINUITY

As is well known the usual notion of *continuous function* in general topological spaces can be sharpened to that of *uniformly continuous functions* in uniform spaces. For quasi-ordoform spaces the corresponding generalized concept will be *quasi-ordoform continuity.*

Definition

A mapping f of a quasi-ordoform space E into another quasi-ordoform space E' will be called *quasi-ordoformly continuous* if for every relation $R' \in \Phi' \in \Sigma'$ (i.e. Φ' is a base of Σ' which is the quasi-ordoform structure, or, more exactly, one of the quasi-ordoform structures, from which the topology of E' may be deduced) there can be found an $R \in \Phi \in \Sigma$ (Φ and Σ have the analogous meaning for E) such that for all $x/y \in R$ also $f(x)/f(y) \in R'$.

Very often $E' = E$, i.e. arguments and values are in the same space. We speak then simply of quasi-ordoform continuity in E.

One has the following easy theorems which are the exact analogies of those for uniform continuity[12]:

1. Every quasi-ordoformly continuous function is continuous.

2. Let f be a quasi-ordoformly continuous mapping of E into E' and g similarly a mapping of E' into E'' (E, E', E'' three not necessarily distinct quasi-ordoform spaces), then the composed mapping $h = g \circ f$ will be a quasi-ordoformly continuous mapping of E into E''.

3.° A 1—1 mapping f between two quasi-ordoform spaces is an isomorphism between them (i.e. an isomorphism of their quasi-ordoform structures, which is more than a homeomorphism of the topological spaces) if and only if f and its inverse f^{-1} are both quasi-ordoformly continuous.

EXAMPLES

I 1. On the rational or real number axis consider the *additive* ordoform structure defined by $0 \leq y - x < \frac{1}{n}$. The family of relations $\Phi = \{A_n\}_{n \in \mathfrak{N}}$ where $A_n = \{x/y \mid 0 \leq x - y < \frac{1}{n}.\}$ and n runs through the set $\mathfrak{N}$ of natural numbers, constitutes an ordoform base of a separate ordoform structure $\mathcal{A}$.

2. Another separate ordoform structure $\mathcal{M}$, the *multiplicative* one, is defined in the same sets without 0, by $\Phi = \{M_m\}_{m \in \mathfrak{N}}$, where $M_m = \{x/y \mid 1 \leq \frac{x}{y} < 1 + \frac{1}{m}\}$. Note that we could also define $0/0 \in \Delta$, but $0/y$ $(y \neq 0)$ in no M_m. Note further that the definition of $\mathcal{M}$ is also of interest in the set of natural numbers (i.e. $E = \mathfrak{N}$).

3. The two structures considered in the same set, say the positive real numbers, are incomparable; even more, whatever the couple of indices n and m we shall never have $A_n \subset M_m$ nor $M_m \subset A_n$. It suffices to state that a) the couple $\gamma_k = (1/k)/[1/k + (1/(n+1))] \in A_n$ but never $\gamma_k \in M_m$ for every m and a sufficiently large k ($k > n$ suffices); b) the couple $\gamma_k = k(m+1)/k(m+2) \in M_m$ but never $\gamma_k \in A_n$ for every n and sufficiently large k (here $k > 1$ suffices).

4. None the less the corresponding deduced topologies, say $\mathfrak{T}_1(\mathcal{A})$ and $\mathfrak{T}_1(\mathcal{M})$ are identical. The family $\mathfrak{A}(x)$ of neighbourhoods $A_n(x)$ of the point x is given by the intervals $A_n(x) = [x, x + \frac{1}{n}]$. The family $\mathfrak{M}(x)$ of neighbourhoods $M_m(x)$ of x is given by the intervals $M_m(x) = [x, x + \frac{x}{m}]$. To every $M_m(x)$ we can find an n such that $A_n(x) \subset M_m(x)$ (by choosing $n \geq \frac{m}{x}$), and, inversely, to every $A_n(x)$ we can find an $M_m(x) \subset A_n(x)$ (by choosing $m \geq nx$). Therefore $\mathfrak{T}_1(\mathcal{A}) \leq \mathfrak{T}_1(\mathcal{M})$ and $\mathfrak{T}_1(\mathcal{M}) = \mathfrak{T}_1(\mathcal{A})$ i.e. $\mathfrak{T}_1(\mathcal{A}) = \mathfrak{T}_2(\mathcal{M})$.

5. But this topology $\mathfrak{T}_1$ is essentially distinct from its dual $\mathfrak{T}_2$ and from the ordinary topology $\mathfrak{T}$ of real numbers. This is exhibited by the fact that in $\mathfrak{T}_1$ every monotonously descending infinite sequence, if bounded below, has a limit, but a monotonously ascending one with upper bound never. In $\mathfrak{T}_2$ the situation is just the opposite, while in $\mathfrak{T}$ both have always limits. $\mathfrak{T}_1$ and $\mathfrak{T}_2$ are incomparable and $\mathfrak{T}$ is just the least upper bound of $\mathfrak{T}_1$ and $\mathfrak{T}_2$.

II 1. Consider the totality Γ_E of *graduations* G (or "hierarchisations" as opposed to "partitions" or "equivalences") of an infinite set E by *finite* ascending sequences of (infinite) subsets E_i, $i = 1, 2, \ldots, n$, i.e. $(\varnothing = E_0 \subset) E_1 \subset E_2 \subset \ldots \subset E_\nu \subset \ldots \subset E_n = E$.

2. Consider the families of graduations $\Gamma = \{G^{(\tau)}\}_{\tau \in T}$, T an index set, such that the symmetric difference of any two distinct set $E_i^{(\sigma)}$, $E_k^{(\tau)}$ is infinite.

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1. As introduced by A. Weil, *Cf.* Bourbaki, *Topologie Générale*, Ch. II. As a matter of fact "uniformisable" space remains "écartisable" (Theorem of Uryson).
2. Directed time is still playing a privileged role in modern quantum mechanics, giving it a certain non-relativistic character.
3. This fact is extensively exploited in our thesis *Monoïdes préordonnés et Chaînes de Malcev*, Paris, 1951.
4. *Cf. l. c.* in ref. 3.
5. Inequalities were already playing an important role in classical mechanics and other parts of physics, long before Heisenberg's inequalities became fundamental in modern physics.
6. *Colloque d'Algèbre et Theorie des Nombres*, Paris, 1949.
7. Iwasawa, K., 1948 (?) (in Japanese). (This incomplete reference was communicated to me by J. E. L. Peck who announced the same theorem in 1950, *Bull. Amer. Math. Soc.*, **56**, 351.)
8. *Cf. l. c.*, in ref. 3 (pp. 18—21).
9. A general binary relation is defined as a subset of the "product set" $E \times F$ (the set of all ordered pairs e/f, $e \in E$, $f \in F$). For our purposes the case $E = F$ suffices. We prefer to write e/f for an ordered pair instead of the more usual (e, f).
10. *A fundamental system of "entourages"* of a uniform structure in the terminology of Bourbaki.
11. *Cf.* e. g. Birkhoff, *Lattice theory*, Ch. I, or Bourbaki, *Theorie des Ensembles* (fasc. des résultats).
12. Bourbaki, *l. c.* in ref. 1.

ADDENDA

After this paper was finished the review of a paper of Konishi, Isao, On uniform topologies in general spaces, 1952, *J. Math. Soc.*, Japan, **4**, 166—188, came to my attention (see 1953, *Math. Rev.*, **14**, 892 –893). It appears to have some points of view in common with us, but to push its exploration farther: the problems of completion and compactification are treated.

SPECTROGRAPHIC OBSERVATIONS ON "NON-ALTERNANT" HYDROCARBONS

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The interest which the fulvenes, systems containing a cross-conjugated system coupled with a five-membered ring

have aroused recently in the circles of both chemists and physicists[1], is due to two facts: (a) they constitute a group of *polar* hydrocarbons (i.e. hydrocarbons exhibiting a finite dipole moment); (b) by the application of the technique of "linear combination of atomic orbitals" (LCAO), one has been able to describe the physical properties of these complicated hydrocarbons with great accuracy; in fact, one could predict a number of features unknown to the organic chemists who had worked with these compounds.

The present study reports on some additional observations concerned with the *spectra* of fulvenes and related compounds, which can all be classified as belonging to the series of "non-alternant" hydrocarbons.

1. ABSORPTION SPECTRA IN CARBON DISULPHIDE SOLUTION

It has been known that the spectra of conjugated polyene hydrocarbons, R. $(CH-CH)_n$. R, are very much dependent on the solvent used in their determination. For dehydrolycopene, e.g., Fieser[2] gives the following wave-lengths of the longest absorption band:

Solvent	λ_{max} (mμ)	Δ (mμ)
Hexane	504	—
Chloroform	528	24
Benzene	531	27
Pyridine	535	31
Carbon disulphide	557	53

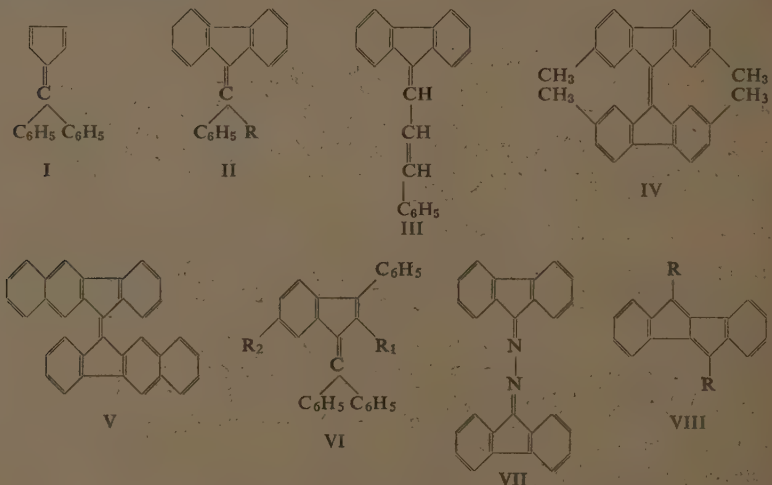
As expected from Kundt's rule, the shift to longer wavelengths is parallel to the increase of the refractive index of the solvent, but the large shift caused by carbon disulphide remains exceptional. Le Rosen and Reid[3] have suggested that the effect is related to the smallness of this molecule which enables it to approach more closely the polyene molecules and thus to exert an unusual polarising effect. As the compounds of the fulvene group are characterised by their high polarisability[4,5], it seemed interesting to compare their spectra in alcohol and carbon disulphide solutions.

TABLE I

Longest absorption band of "fulvenoid" molecules in alcohol and carbon disulphide

Substance	λ_{max} *in* mμ (log ε) in alcohol	in carbon disulphide	Δ (mμ)
1. Diphenylfulvene (I)	328 (4.36) [6]	375 (3.40)	47
2. Phenyl-dibenzofulvene (II, R=H)	326 (4.17) [6]	375 (3.70)	49
3. Styryl-dibenzofulvene (III)	380 (4.67) [6]	420 (4.54)	40
4. Diphenyl-dibenzofulvene (II, R=C_6H_5)	338 (4.14) [6]	375 (3.92)	37
5. 2,7,2',7'-Tetramethyl-dibiphenylene-ethene (IV)	459 (4.34) [7]	475 (4.34)	16
6. 2,3,2',3'-Dibenzo-dibiphenylene-ethene (V)	530 [8]	550	20
7. 1-Benzhydrylidene-2-chloro-3-phenyl-indene (VI, R_1 = Cl, R_2 = H)	370 (4.10) [5]	380 (4.20)	10
8. 1-Benzhydrylidene-2,6-dibromo-3-phenyl-indene (VI, R_1 = R_2 = Br) [9]	366 (4.28)	385 (4.20)	17
9. Fluorenone-ketazine (VII)	361 (4.26) [5]	375 (4.20)	14
10. Dimethyl-diphensuccindadiene (VIII, R = CH_3)	412 (4.01) [10]	421 (4.26)	9

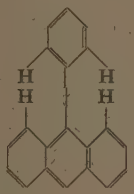
It is seen that the fulvenes proper (No. 1—4) show the same very large bathochromic shift in carbon disulphide solution as the polyenes; it is equally interesting that the spectra of the dibiphenylene-ethylene derivatives (IV, V) which are outstanding for their high polarisability in spite of the absence of a dipole moment[5] are also substantially affected by this solvent.



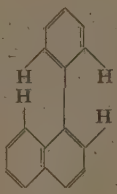
2. THE ABSORPTION SPECTRA OF 9,12-DIARYL-DIPHENSUCCINDADIENES (VIII, R=ARYL).

Phenyl groups in the 9- and 10-positions of the anthracene and in the 1-position of the naphthalene system do not cause a bathochromic shift in the absorption spectra of the parent hydrocarbons, whilst, e.g., a phenyl group added at the end of an open-chain conjugated system is equivalent in bathochromic power to 1.5 double bonds[11,12], and whilst 2-phenyl-naphthalene shows a similar bathochromic shift[13].

This effect has been ascribed[14] to the steric interference of the ortho-hydrogen atoms in, e.g., 9-phenylanthracene (A) and 1-phenyl-naphthalene (B) which prevents the phenyl rings from orienting themselves in the same plane as the anthracene and naphthalene system, respectively, and therefore from creating the — planar — conjugated system which is responsible for the bathochromic effects. Pullman, Berthier and Baudet[15] have calculated recently that the introduction of phenyl groups into the systems of "non-alternant" hydrocarbons can have both a hypsochromic or bathochromic effect, depending on the charge distribution in the parent systems and the mutual polarisabilities between the atoms of the molecules; in this case, the hypsochromic effect is independent from steric interference with planarity. It appeared interesting to study the 9,12-diarylated diphensuccindadienes (VIII, R=aryl) (Table II). Methylation has a slightly hypsochromic effect on the wavelength of the longest absorption band of the parent hydrocarbon, as predicted by the theory[16], whilst introduction of phenyl, *m*- and *p*-tolyl or *p*-methoxyphenyl groups has a pronounced bathochromic effect, although the steric interference in (C)



A



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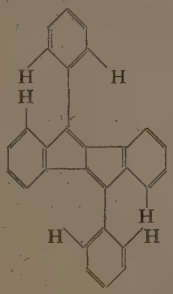


C

would not be smaller than in 1-phenylnaphthalene (B). Pullman[17] has, indeed, applied his calculations to the case of the 9,12-diphenyl compound (VIII, R = C_6H_5) and has arrived at a correct prediction of the observed effect; the present case is unique in that the *bathochromic effect is due to the lack of coplanarity*; a planar structure of this particular type of molecule would probably cause a hypsochromic shift.

TABLE II

Longest absorption band of the 9,12-*diaryl-diphensuccindadienes* (VIII, R=*aryl*) (*in alcoholic solution*)

Substance		λ_{max} in mμ (log ε)
Diphensuccindadiene	(VIII, R = H)	415 (4.18)[18]
Dimethyl-	(VIII, R = CH_3)	412 (4.09)[10]
Diphenyl-	(VIII, R = C_6H_5)	446 (4.23)[19]
Di-*m*-tolyl-	(VIII, R = 3-$CH_3.C_6H_4$)[20]	460 (4.20)[a]
Di-*p*-tolyl-	(VIII, R = 4-$CH_3.C_6H_4$)[21]	450 (4.22)[b]
Di-*p*-methoxyphenyl-	(VIII, R = 4-$CH_3O.C_6H_4$)[22]	448 (4.20)[c]

Notes: a) other bands: 264 (4.56), 342 (4.03), 435 (4.21).
b) other bands: 263 (4.68), 323 (4.22), 426 (4.22).
c) other bands: 262 (4.68), 285 (4.50), 425 (4.22).

It should be noted that the introduction of the – auxochromic – methoxyl group has practically no effect in this series.

3. INFLUENCE OF HALOGEN SUBSTITUTION ON THE SPECTRUM OF "NON-ALTERNANT" HYDROCARBONS

It has recently[23] been shown that introduction of halogen into the 1-position of the azulenes, which belong to the above-defined class of hydrocarbons, has a pronounced bathochromic effect. In the fulvene series, only qualitative observations of the same type have so far been made[9], and it seemed interesting to investigate a few pertinent cases quantitatively. Table III shows that halogenation of the 2-position in 1-benzhydrylidene-3-phenyl-indene (VI; $R_1 = R_2 = H$) also causes a bathochromic shift of the longest absorption band. In the diphensuccindadiene series, the effect is much less in evidence. No theoretical explanation for this phenomenon has yet been given. It may be interesting to recall that halogenation of the *benzene* rings in the dibenzofulvenes has a negligible influence on their absorption spectra[24].

TABLE III

Influence of halogen substitution on the longest absorption band of some "non-alternant" hydrocarbons (in alcoholic solution)

Substance	λ_{max} in mμ (log ε)
Azulene	700[23]
1-Chloroazulene	737[23]
1-Bromoazulene	740[23]
1-Benzhydrylidene-3-phenyl-indene (VI, $R_1 = R_2 = H$)	350 (4.02)[25] a)
1-Benzhydrylidene-2-chloro-3-phenyl-indene (VI, $R_1 = Cl$, $R_2 = H$)	370 (4.18)[5]
1-Benzhydrylidene-2,6-dibromo-3-phenyl-indene (VI, $R_1 = R_2 = Br$)	366 (4.28)
Diphensuccindadiene (VIII, R = H)	415 (4.18)
9.12-Dichloro-diphensuccindadiene (VIII, R = Cl)	420 (4.28)

Note: a) other bands: 250 (4.25), 299 (3.92).

4. THE SPECTRUM OF 2,3-DIARYL-INDONES (X)

Coan, Trucker and Becker[26] and Shapiro and Becker[27] have studied the spectra of a large number of 2,3,4,5-tetraaryl-cyclopentadienones (IX), and have interpreted their findings by the assumption that of the two main bands in these spectra the shorter one is due to the system IXa, the longer one to the system IXb. It should be possible to verify this simplifying hypothesis in a less complicated system such as that of the 2,3-diaryl-indones (X, Xa, Xb), in which the possible influence of the "branching" of the conjugation, involving the aryl groups in positions 4 and 5 of (IX), is eliminated. Table IV summarises the results.

TABLE IV

Absorption spectra of 2,3-diaryl-indones (X) *(in alcohol)*

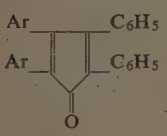
Substance	λ_{max} *in* m μ (log ε)		
1. 2,3-Diphenyl-indone[5]	234 (4.48)	256 (4.50)	440 (3.24)
2. 2-Phenyl-3-(*p*-methoxyphenyl)-[28]	268 (4.49)	322 (3.78)	450 (3.42)
3. 2-(*p*-Methoxyphenyl)-3-phenyl-	224 (4.76)	291 (4.23)	435 (3.16)
4. 2-Phenyl-3-(*p*-bromophenyl)-[27]	260 (4.58)		436 (3.30)
5. 2-(*p*-Bromophenyl)-3-phenyl-	265 (4.59)		440 (3.36)
6. 2-Phenyl-3-(α-naphthyl)-[28,29]	223 (4.84)	264 (4.55)	416 (3.53)
7. 2-(α-Naphthyl)-3-phenyl-	224 (4.80)	286 (4.28)	432 (3.18)
8. 2-Phenyl-3-bromo-[39]	245 (4.54)	296 (3.87)	425 (3.08)

The results cannot easily be accounted for by the assumption that the two longest bands in the spectrum of 2,3-diphenylindone correspond to the absorption paths Xa and Xb. Introduction of a methoxy group into the para-position of the 3-phenyl indeed shifts the 256 m μ band of 2,3-diphenylindone considerably, but the analogous operation in the 2-phenyl also affects only the 256 m μ band and has no influence on the 440 m μ band. Bromination of either the 2- or 3-phenyl has no effect on the latter band (the 256 m μ band disappears in these two cases). Replacement of the phenyl groups by α-naphthyl radicals has a hypsochromic effect on the 440 m μ band and a bathochromic effect on the 256 m μ band, in both cases. It should be noted that replacement of the 3-phenyl group by a bromine atom has a bathochromic effect on the 256 m μ band — as the above-mentioned hypothesis would predict — but a hypsochromic effect on the 440 m μ band which is not altogether in accord with that hypothesis. It appears that the assignment of the two bands of 2,3-diphenylindone to two distinct absorption paths is not feasible.

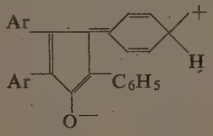
The description of the substances studied — as far as they are new — will be published elsewhere.

ACKNOWLEDGEMENT

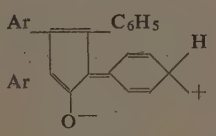
The author is indebted to Hannah Weiler, Scientific Department, Israeli Ministry of Defence, for the determination of the absorption spectra.



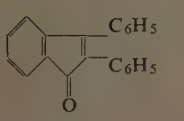
IX



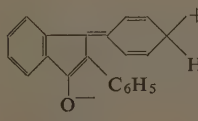
IXa



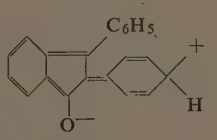
IXb



X



Xa



Xb

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ISRAEL PHYSICAL SOCIETY

THE FOUNDING MEETING

April 12—13, 1954

General and Industrial Physics (Monday afternoon)

The Problems of Cavitation in Pipelines after Cessation of Pumping, MOSHE BITON, *Hydraulic Engineer, Tel Aviv.*

Indefinite Admittance Representation of Linear Networks. JACOB SHEKEL, *Scientific Department, Ministry of Defence.* Kirchhoff's Rules, that form the basis of network analysis, apply to networks composed of two-terminal branches. Two postulates are now proposed, applicable to networks whose basic elements may have more than two terminals (e.g., multi-electrodes vacuum-tubes or transistors). The resulting method also makes possible analysis without specifying the voltage-reference terminal.

There is some redundancy in the definition of the currents, and excessive degrees of freedom in the voltages; but it is shown that these two effects compensate and result in no ambiguity in the definitions of instantaneous power and of the admittance matrix.

The various points raised are interpreted geometrically as relations in n-dimensional Euclidian space.

Scattered Solar Radiation. Its Computation and Measurement, N. ROBINSON, *Solar Radiation Laboratory, Technion, Israel Institute of Technology, Haifa.* The scattered solar radiation is a part of the total solar energy radiation created by the influence of the atmosphere. This is important because of the scattered ultraviolet and the heating effect. Because of its undirected character it is impossible to compute by means of the solar coordinates. Only the influence of pure air on the spectral parts of solar radiation can be precisely computed. The various particles present in the atmosphere complicate computation, especially since the quality and quantity of these particles change rapidly.

Therefore, it is only possible to measure the scattered radiation separated from the total and sky radiation.

The author has developed an auxiliary instrument for this purpose.

The instrument is based on the principle of permanent shadowing of the pyrheliometer, and the shadowing device may be adjusted according to the date. The design of this instrument allows the width of the shadow cast on the pyrheliometer to remain constant all the year round. For this purpose the pyrheliometer is in the centre of a sphere, and its surface rests in the plane of the hemisphere. A strip was cut from the hemisphere having a radius ten times greater than its width. This strip is movable, according to the solar coordinates. The various strips are computed for the dates required.

On the Calculation of the Detonation Parameters of some Explosives, Y. MANHEIMER, *Scientific Department, Ministry of Defence.* The pressure, temperature and detonation velocity of an explosive as well as the material velocity and the specific volume of the detonation products can be calculated from the known loading density and chemical composition of the explosive.

The first step is the determination of the detonation products, which can be obtained from a number of equilibrium conditions, dependent on the unknown temperature and pressure. An equation of state for the detonation products must be assumed, since they are far from ideal gas conditions; this is done by evaluating the second, third, fourth and fifth virial coefficients. The calculation is performed according to the hydrodynamic theory of detonation (Rankine-Hugoniot equation, Chapman-Jouguet condition), employing a method of successive approximations[1].

The calculation was carried out for mixtures of tetranitromethane and toluene[2] of different composition (this is a powerful liquid explosive known as panclastite) and we obtained the result that the maximal pressure and detonation velocity were obtained for a composition of 15% toluene and 85% tetranitromethane.

Measurements of the intensity of the shock wave issued by the detonation of these mixtures

showed good agreement with the theoretical results. In this case the explosive is close to stoichiometric composition and it was therefore possible to neglect the equilibrium equations (taking these equations into account alters the result by less than 1%).

Another explosive for which the calculations were carried out is pure mononitromethane, which has a large negative oxygen balance, and here we had take into account the equilibrium coefficients. The detonation velocity obtained as a result of the calculations was in agreement with measurement within 5%.

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Some Measurements of Detonation Velocities with Electrical Conduction Probes and Spiral Sweep Oscillogram, G. NAHMANI, *Scientific Department, Ministry of Defence.* Velocities of detonation of various explosives were measured by using the electrical conduction due to ionization in the detonation wave to short circuit gaps in probes placed in the explosive and thereby provide timing signals. The signals were recorded on an oscillograph with a single spiral sweep, of about 10-15 turns and a frequency of 100 KC, by blanking the beam for about $1/2$ microsecond each time.

The accuracy with which velocities may be measured by this method is high and is limited in practice to the accuracy with which distances between the probes are measured. The method does not suffer, as direct photographic methods do, from masking effects by illumination from shock waves and it is useful for measuring velocities in cased explosives or inside charges.

Two examples are given which illustrate the type of work done by this method.

The detonation velocities of a solid propellant in a powdered form were measured by placing the powder in cardboard tubes, threaded by fine copper-wire probes, at 2 cm imtervals, and initiating with No. 10 detonator. As expected the velocity was found to vary with the grain size, increasing when the powder was finer.

Nitromethane with no confinement failed to detonate in contact with a 30 gram charge of PETN. Under strong confinement, however, it did detonate. The liquid was poured into a 2.5 cm dia. hole drilled in a 6 cm dia. steel shaft and 7 probes were placed inside at 2 cm intervals. A constant velocity of 6200 m/sec was recorded.

On the Measurement of Transient Pressures and Synchronizing Devices, M. BIRK, *Scientific Department, Ministry of Defence.* Transient pressures of 10—10^4 atm. and 10^{-3}—1 sec. duration have been measured by gauges of the following types: resistive, piezoelectric and capacitive. The first method has been found most convenient for routine work, there being no need for high insulation of gauge and cable, whereas in the second method trouble is sometimes caused by leakage, especially when the relative humidity is high. On the other hand the piezoelectric method has the advantage of giving easily recorded signals generated by pressures as low as one atmosphere. The capacitive method requires a r.f. oscillator near the place where the pressure is measured, which is sometimes a disadvantage.

The recording instrument in the described methods is a cathode-ray-oscillograph and the measured processes should be synchronized with the camera or the beam. When the process is electrically initiated, synchronization is achieved by using a multi-contact relay which triggers a camera shutter and then closes the ignition circuit, the beam continuously running on the scope. If a drum-camera is used, the drum synchronizes beam-brightness with ignition. However, in fast processes single-sweep operation had to be used. A variable delay between beam-triggering and ignition was obtained by a R-C network firing a thyratron, which was thus used to close the ignition circuit at the desirable moment.

A monostable multivibrator which brightened the beam for a desired time proved efficient in some operations.

In cases of mechanical initiation the problem of synchronization had to be solved for each case separately. Generally, the movement of some part was used to close or to open an electric circuit, thus triggering the beam.

The Solution of the Problem of the Kaleidoscope, A. MARKEL, *Natanya.* The formjulae given in the literature regarding the number of images obtained by a kaleidoscope (or by two plane mirrors nclined at an an angle j with each other) are either5 incorrect or at best not exact; for this reason some books on optics do not mention this problem at all. An accurate solution is suggested below:

To the angle of inclination **0** whose arc **k0** is obtained by dividing the perimeter of the cirrcle **p** by a whole number, corresponds a regular polygon whose number of sides is equal to the divisor in such a way that two consecutive symmetrical axes of the polygon coincide with the arms of the angle of inclination, and one vertex of the polygon is situated on the bisector of the angle. The vertices of the polygon which are symmetrical alternately to the axes (according to the arrangement of the points symmetrical to the object inside the angle **0**) may be arranged in two series whose sequence and number of rterms detemines the sequence and number of images. The arrangement of the series is such that if the image falls on the arms of the vertical angle or inside it, the series is not continued. (This implies reduction of the number of terms).

This method may be used for cases in which the

arc of the angle of inclination is obtained by dividing theperimeter of the circle by an improper fraction. Determination of the exact number of imges is made possible only by use of regular polygons and cyclic groups of the vertices.

A Summary Account of the Origin of the Function and the Variable before Galileo, S. HEVESI, *Museum of Modern Art, Haifa.* An essential part of Galileo's laws, the variable and its correlated use, are shown to have been evolved through centuries from the discussion of the intensity of qualities by scholastic philosophers. Not only was the new science different in nature from the static mechanics of the Greeks, but the raising of this problem of change in intensity already meant a radical departure from Greek ways of thought. The reception of Aristotelian philosophy into an alien linguistic is suggested as the stimulus for the elaboration of the new conception of change.

Theoretical Physics (*Monday afternoon*)

Some New Levels in the Spectrum of Yb II, G. RACAH, *The Hebrew University of Jerusalem.* Constant differences between known levels and unclassified lines were looked for by a punched-cards technique; coincidences were obtained, which suggest the existence of 19 levels which are not comprised in the unpublished list communicated by W. F. Meggers. A definitive confirmation of the existence of these levels may be obtained only by examining the Zeeman data of the corresponding lines.

A group of suggested levels agrees with the values calculated by S. Aberbach and D. Sperber for the configurations $f^{13}sd$ and $f^{13}sp$; the position of the other group agrees with the expected position of the low levels of $f^{13}d$[2]. The intensities of the transition confirm the assignments of the levels to these two groups.

The Spectrum of Yb II*, D. SPERBER, *Technion, Israel Institute of Technology, Haifa.* The configurations $f^{13}sp$ and $f^{13}sd$ of Yb II have been calculated, and the configuration $f^{13}pd$ of the same element is now being calculated. Use was made of the values of the energy levels, the angular momenta J and the Lande g-factors communicated by W. F. Meggers. jj-coupling between the outer electrons and the missing electron was assumed. LS-coupling was assumed between the two outer electrons. These assumptions were based on the positions of the levels, the Lande g-factors, and the intensities observed. The linear equations for the energies were formed according to this scheme. The parameters appearing in these linear equations were initially determined by the method of least squares, and the use of known levels, to a first approximation. Closer approximations to these parameters were then made by the use of perturbation theory and numerical diagonalization. The equations for the unknown levels were then solved via the use of the best parameters determined above. These solutions agreed with the values arrived at for these levels by an experimental method used by Racah.

* This work was done at the Hebrew University of Jerusalem under the direction of Prof. G. Racah.

Statistical Geometry and Quantum Theory, NATHAN ROSEN, *Technion, Israel Institute of Technology, Haifa.* Since there exists a limitation on the accuracy of measurement of position due to the existence of a fundamental length associated with the elementary particles, it appears necessary to introduce the concept of statistical geometry in which, corresponding to a given measured position, there exists a distribution function for the true position[1]. On the basis of the homogeneity and isotropy of space and the assumption of the independence of the Cartesian coordinates one arrives at a distribution function in the form of a Gaussian error function. From the standpoint of the quantum theory, one can associate with each point of space both a set of coordinates and a set of momenta, and it is plausible to characterize each point by means of a set of non-Hermitian variables, each of which has as its real part a coordinate and as its imaginary part (to within a constant dimensional factor) the conjugate momentum. The eigenvalues of these variables are continuous, and the eigenfunction is a Gaussian error function, so that the probability distribution is of the same form as above. These ideas can be extended and modified to conform to the requirements of the relativity theory. If one assumes that all fields are described by components which are functions of these new variables, then it is possible to choose a representation in which the coordinates are diagonal, but the fields are nonlocalized. This leads to the removal of divergences, such as the infinite self-energy of the electron.

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Investigations on the Integral Equation Method and the Assumption of Superposition in the Theory of Condensed Systems, J. MAZUR and J. E. MAYER, *The Weizmann Institute of Science, Rehovot* and *The University of Chicago, Chicago, Illinois.*

1. A set of molecular distribution functions F_n have been defined which depend on molecular coordinates and on parameters which describe the

thermodynamic state of the system. The parameters chosen are the temperature and the activity. The activity becomes identical with the molecular density at an infinite dilution.

2. The distribution functions which describe two systems at different activities but at the same temperature, are interrelated through an integral equation, which was derived from the Grand Canonical Ensemble of Gibbs.

3. A potential of average forces of n particles has been defined as a quantity which is proportional to the logarithm of F_n. For zero activity the potential of average forces becomes identical with the classical potential energy of intermolecular forces.

4. It has been assumed that, similarly to the potential energy of intermolecular forces, the potential of average forces can be expressed as a sum of interactions between all possible molecular pairs. This assumption has been called the superposition assumption.

5. Two reciprocal integral equations have been constructed, which have as the unknown quantity the potential of average forces. Using the superposition approximation we transform one of these two equations into Fredholen integral equation, whose solution enables us to test the validity and accuracy of the superposition assumption. Generally, there is a mathematical inconsistency in the assumptions of superposition in both the potential energy of intermolecular forces and the potential of average forces, which can be shown by symmetrizing the kernels of the two reciprocal integral equations and expanding them in orthogonal sets.

6. As a simple case, a liquid with a hard-sphere potential energy for the intermolecular forces has been investigated. The apparent phase transitions which this system seems to possess, arise from the incorrectness in the superposition approximation.

Two-Component Wave Function, S. BURSZTEIN, *Haifa.* A two-component first-order wave equation is studied for a neutral particle of vanishing rest-mass (neutrino). It is shown that there may exist two kinds of neutrinos with opposite orientations of spin relative to momentum (right and left particles). Some results concerning the "Zitterbewegung" of Schroedinger are deduced. The possibilities of an extension of the theory to charged particles is considered and some possible consequences regarding asymmetry in nature are discussed.

Multiple Meson Production and Isotopic Spin Conservation. Y. YEIVIN, *The Atomic Energy Commission, P.O.B. 7056, Hakirya.* Fermi's theory of pion production[1] enables one to calculate the (relative) cross-section for production of p pions in a nucleon-nucleon collision as a function of the collision energy. The theory was generalized[2,3] so as to include production of heavy mesons (K-particles) as well, and give the cross-section for production of k heavy mesons and p pions, and also the ratio n_K/n_π of the average number of heavies to that of pions produced in a collision. This ratio can be compared with experimental results.

While Fermi deals with classical, independent particles, Haber-Schaim et al. consider the different possible charges of the particles appearing after the collision, consistent with charge conservation, and also the possible identity of some of them. In a recent paper[4], Fermi considers the different charge possibilities consistent not only with charge, but also with isotopic spin conservation.

Our purpose is (a) to give general formulae for the corrections of the production cross-sections which follow from charge (and isotopic spin) conservation, and possible identity of some of the particles; and (b) to show that there is hardly any difference — as far as meson production is concerned — between charge and charge plus isotopic spin conservation.

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Exciton Band Structure in the Benzene Crystal, D. FOX and O. SCHNEPP, *Technion, Israel Institute of Technology, Haifa.* Non-conducting excited states of insulators can presumably be described in terms of "excitons", or excitation waves. Each excited level of the free molecules is replaced, in the crystal, by a band, with each level of the band characterized by the exciton propagation vector. We have calculated the energy as a function of the propagation vector for the two lowest bands of the benzene crystal. These bands correspond to the molecular states belonging to the representations $B_1\mu$ and $B_2\mu$, respectively, of the molecular point group D_{6h}. It was found that in an expansion of the Coulomb interaction potential between molecules, the octupole-octupole terms are the lowest order contributions to the spread of the band. Calculations give the width of the $B_1\mu$ band as 1500 cm^{-1}, of the $B_2\mu$ band, 65 cm^{-1}. It may be shown that there are at most three levels of each band which give electric dipole transitions to the ground state. Two such transitions originating in the lower band have been observed. The separation between them is 25 cm^{-1}; the predicted value, on the assumption that the lower molecular state belongs to $B_2\mu$, is 20 cm^{-1}.

Linear Relations in Crystalline Potential Problems, P. NAOR, *Technion, Israel Institute of Technology, Haifa.* The calculation of crystalline potentials, lattice sums and Madelung constants is dealt with from the view point of space group theory. It is assumed

a) that the potential emanating from any single atom in the crystalline assembly is isotropic with respect to this atom, and

b) that no distortion—not even isotropic—of such atomic potential takes place on expanding or contracting the crystalline assembly.

Introducing the crystallographic concept of lattice complexes it is shown that these assumptions lead to two types of relations between different crystalline potentials — a) Reciprocity relation and b) Additivity relation. These two concepts may be used for the final formulation of linear dependence between crystalline potentials, lattice sums and Madelung constants.

He II: Kinetics of Superflow in the Bose-Einstein Gas Model, P. R. ZILSEL, *Technion, Israel Institute of Technology, Haifa.* The Bose-Einstein condensation mechanism can account for the existence of separate hydrodynamic velocity fields for the normal and the superfluid, if one assumes a suitable single-particle energy spectrum. The microcanonical gas distribution for nonzero total momentum P is

$$n_j = g_j \,/\, \{ \exp[(\epsilon_j - \varkappa P_j - \mu)/kT] - 1 \}$$

where

$$\varkappa = (\partial E/\partial P)_{S,N.}$$

Below the condensation temperature, if the energy spectrum has a sharp minimum (gap or cusp), only the excited part of the gas contributes to the total momentum, the condensate remaining "frozen" in momentum space. This rigidity of the condensate in momentum space plays the same role as the rigiditv of the superelectrons on imposition of a magnetic field in the London theory of superconductivity. P acts as an additional thermodynamic variable, states with $P \neq 0$ being macroscopically metastable and corresponding to relative motion of the two fluids with velocity v. The hydrodynamic equations of the two-fluid theory follow directly from the statistical model. The basic hydrodynamic assumptions of the two-fluid theory are thus reduced to an assumption concerning the form of an effective single-particle spectrum, and the parallelism between the theories of superconductivity and superfluidity is clearly exhibited. The present model permits, in particular, the introduction of the mathematical form of Landau's phonon and roton spectra within the framework of the Bose-Einstein condensation picture.

General Session (Monday evening)

Opening of the first meeting of the Israel Physical Society by its first president, Prof. Giulio Racah

The Influence of Modern Physics on Organic Chemistry, E. D. BERGMANN, *Scientific Department, Ministry of Defence. Tel Aviv.*

The Role of Physics in Industry, M. CHWALOW, *Technion, Israel Institute of Technology, Haifa.*

Atomic, Nuclear and Molecular Physics (Tuesday morning)

Search* for Radiations from Pb^{205}, C. D. CORYELL**, R. H. HERBER, T. T. SUGIHARA and W. E. BENNETT, *Department of Chemistry and Laboratory for Nuclear Science, Massachusetts Institute of Technology, Cambridge, Mass. U.S.A.* The unidentified nuclide Pb^{205} is expected to decay by electron capture to Tl^{205} with 0.4 Mev decay energy. Its absence in nature indicates an upper limit of 3×10^8 years on the half-life. De Shalit and Goldhaber (1953) have pointed out that the transition (expected to be $P^1/_2\, S^1/_2$) should be "superslow" because of departures from the strict single-particle model.

Predicted amounts of Pb^{205} have been synthesized by the (d,p) reaction on natural lead with the M.I.T. cyclotron and by the (n, γ) reaction

* This work was supported in part by the U. S. Atomic Energy Commission.

** Louis Lipsky Fellow, Weizmann Institute of Science, Rehovot, 1953—54.

in the Arco MTR pile with 0.4 g of Pb enriched to 26% Pb^{204} from the Stable Isotopes Division of the U.S.A.E.C. All attempts to find *K* x-rays of thallium (73 Kev) from its decay have been fruitless. Using the thermal neutron cross-section of Pomerance (1952) and the integrated flux of 3×10^{20} n/cm^2, it is shown that the half-life for *K*-electron capture is outside the range 10 days to 10^{10} years. Studies made immediately after such bombardments eliminate *K*-capture half-life in the region seconds to 10 days. Thus it can be said that Pb^{205} does not decay appreciably by *K*-electron capture, and search is being turned to the detection of *L* and softer x-radiation.

Description of a β-ray Spectrometer, J. BURDE, *Department of Physics, The Hebrew University of Jerusalem.* A thin lens magnetic β-ray spectrometer has been built, for the measurements of the energies and intensities of nuclear radiations from radioactive substances.

The axis of symmetry of the instrument is vertical. The distance between source and detector is one metre. The vacuum chamber is a brass tube of outer diameter 17.5 cm. The axially symmetric magnetic field is produced by an iron-free short solenoid surrounding the vacuum chamber built in the form of eight concentric coils of copper wire. Cooling is provided by passing coolant through 3 layers of copper tube which lie between the coils. The solenoid contains 180 kg of copper. The vacuum chamber contains a system of aluminium baffles which fix the transmission and resolution of the instrument. The transmission can be changed externally by moving a baffle via a Wilson vacuum seal. A vacuum lock and gate have been incorporated so that the course can be changed without destroying the vacuum in the main chamber. An operating pressure of 10^{-5} mm is used.

Tests of the performance of the instrument have been carried out using the thorium *F* line. The position of the "ring focus" (which arises from spherical aberration) has been determined by direct photography, using a 10 millicurie recoil source of thorium B and a four hour exposure. The photographs show excellent axial symmetry. By placing an annular baffle of suitable annular radius at the ring focus, it is hoped to obtain a resolution of 0.6% for a transmission of 0.6%. With magnet cooling, the spectrometer should be capable of focussing electrons up to energies of 5 MeV.

Single and Complex Probability Functions in the Counting of Alpha Particles, H. LINDEMAN, *Technion, Israel Institute of Technology, Haifa.* When alpha particles from a steady source of radioactive substance (for instance polonium) are randomly emitted, then the probability that a certain number of particles will be counted in a given time interval is given by the Bateman-Poisson law.

In certain cases of radioactive disintegration, however, the alpha emission does not occur randomly with respect to time. This has been pointed out by Adams, who takes as example the thorium series. Here, if we take 5 min. as the length of the counting interval, the mean life of thoron and thorium A is short in comparison with this length. Therefore, the probability that a certain number of alpha particles will be counted in a given interval of time is given by a complex probability function which differs considerably from the (single) Bateman-Poisson function.

Measurements on the probability function of the disintegration of polonium, the radium series and the thorium series were carried out with a methane flow proportional counter. The measurements showed that disintegrations of polonium and the radium series occurred in good agreement with the Bateman-Poisson law, while those in the thorium series obey a complex probability function, as should be expected on the basis of the theoretical work of Adams.

Absorption and Scattering of Microwaves by a DC Discharge in Helium and air, Z. GELLER and W. LOW, *Department of Physics, The Hebrew University of Jerusalem.* Microwave radiation is scattered and absorbed by electrons in a discharge. The magnitude of these effects is a function of the electron density and collision frequency of the electrons.

We have studied changes in transmission and scattering at an angle 90° of microwaves by a DC discharge. The discharge tube of 50 cm length and aluminium electrodes of 5 cm diameter was subjected to 1 and 3 cm radiation generated by a low powered klystron. The transmitted and scattered waves were detected by means of a crystal detector. Detection was also made at 90° polarization. Considerable difficulties were experienced by diffraction and interference effects.

We have measured the attenuation and scattering as a function of gas pressure and current (up to 30 ma.) at various places along the discharge. From the preliminary results the following conclusions can be drawn.

1. The magnitude of the attenuated and scattered waves varies along the discharge.
2. At constant pressure and current the attenuation and correspondingly the scattering are very large in the negative glow region. They go through a distinct maximum near the position of the reversed electric field.
3. In the Faraday dark space there is a considerable decrease of attenuation and scattering from the high values in the negative glow, falling to a nearly constant but low attenuation and scattering in the positive column.
4. For a given pressure similar regions in the discharge show larger attenuation and scattering for larger current densities.

Effects of Electric Fields on the Luminescence of the Phosphor SrS:Eu,Sm, I. T. STEINBERGER, E. ALEXANDER, A. E. BRAUN, W. LOW and L. WEIDENFELD, *Department of Physics, The Hebrew University of Jerusalem.* The effect of application of alternating electric fields during various stages of the excitation and the exhaustion on this infrared-stimulable phosphor were investigated. The experimental arrangement was similar to that used by Destriau[1] in his research on electroluminescence.

The following main effects were observed:

An A.C. electric field (of the order 10^4 V/cm, 50 c.p.s.) acting during the excitation by blue light, causes a momentary increase of illumination and a subsequent stationary quenching of the fluorescence. On switching off the voltage during the excitation a second momentary illumination results. Furthermore, the field which acted during the excitation increases both the phosphorescent and the infrared-stimulated brightness. The increase of infrared-stimulated emission persisted even if several hours had elapsed between the end of the excitation and the beginning of stimulation.

On the application of an electric field during phosphorescence or during the infrared-stimulated luminescence a momentary increase of illumination results with no subsequent quenching of the luminescence. This momentary illumination grows with the field strength, and decreases with the decay of the luminescent emission.

The effects are strongly interrelated. For example, if the field acted during the excitation and thus caused an increase of the subsequent infrared-stimulated emission, a second switching-on of the field during the stimulation will cause only a very feeble momentary increase of illumination.

An attempt will be made to explain these results according to the various models on electroluminescence.

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Colour Centres in Alum Crystals, E. A. BRAUN and A. MANY, *The Hebrew University of Jerusalem,* A. E. SIMCHEN, *Scientific Department, Ministry of Defence.* We have studied the production of colour centres by X-ray irradiation in the following alum single crystals:

1. $KCr(SO_4)_2, 12H_2O$; 2. $KAl(So_4)_2, 12H_2O$;
3. $NH_4Al(SO_4)_2, 12H_2O$; 4. $NH_4Fe(SO_4)_2, 12H_2O$.

The specimens were exposed to X-rays for various periods of time, and their absorption spectra were measured before and after irradiation.

Of these substances, colour centres are produced in chromium potassium alum only. An additional absorption band at 228 mμ was found; the measured density of this band increases up to a saturation value with the time of X-ray irradiation. At room temperature, this band decays slowly (several weeks), while at liquid air temperature no decay is observed. The absorption band can be bleached by exposure to ultraviolet light of any wavelength within the band.

This proves that the absorption band arises from colour centres of a single energy level.

Further work is now being carried out, and a full account will be published in due course.

Electron Microscopic and Diffraction Studies of Some Paraffinic Sodium Soaps[1], M. L. E. CHWALOW[2], *Temple University, Philadelphia, Penna., U.S.A.* Electron microscopic investigations of the crystalline structures formed by the paraffinic sodium soaps, from sodium butyrate ($NaC_4H_7O_2$) through sodium stearate ($NaC_{18}H_{35}O_2$), show these to consist, primarily, of ribbon like fibres, which in the higher molecular weight soaps occasionally form closed rings. The maximum widths of these fibres decrease as the molecular weights of the soaps increase. As is known, sodium soap molecules align themselves with the long axes of their hydrocarbon chains parallel and with their ionic ends adjacent to each other, thus forming bimolecular laminae. Present electron diffraction studies substantiate this and demonstrate that the long axes of the hydrocarbon chains are oriented essentially normal to the planes of the above noted ribbon like fibres. Knowledge of this plus analysis of the van der Waals forces acting between the adjacent hydrocarbon chains and the forces prevailing in the ionic layer, lead to a direct explanation of the structures and structural changes observed electron microscopically.

1. Materials and equipment (RCA) were provided by Socony Vacuum Laboratories.
2. Present Address — Technion, Israel Institute of Technology, Haifa, Israel.

The First Absorption System in the Ultraviolet of Crystalline Hexamethylbenzene, O. SCHNEPP, *Technion, Israel Institute of Technology, Haifa,* and D. S. MCCLURE, *University of California, Berkeley, California.* The absorption spectrum of single oriented crystals of hexamethylbenzene in the near ultraviolet in polarized light has been photographed on a 3 metre Littrow quartz spectrograph, at 20°K, and the emission corresponding to the lowest absorption system was photographed under the same conditions on a medium quartz spectrograph. The crystal structure allows unambiguous separation between the polarization components perpendicular and parallel to the plane of the benzene ring. The results show that the transition which corresponds to the 2600 A transition of benzene, contains a well developed system polarized perpendicular to the plane of the ring, in addition to a more intense in-plane system. A vibrational analysis was made and showed that the 0—0 band occurs only in the perpendicular component and totally symmetric

vibrational progressions are superimposed on it. The in-plane component, on the other hand, represents a forbidden transition made allowed by a vibration analogous to a molecular vibration of symmetry E_{2g} of the benzene molecule. This vibration has a frequency of 450 cm^{-1} in the ground state as measured on the emission photographs, and this is in excellent agreement with an observed Raman frequency of the molecule. The spectrum can be explained on the assumption that the hydrogen atoms of the methyl groups perturb the π-electron system and allow the mixing in of Rydberg states. It is also necessary to assume that the methyl groups are in a state of interlocked rotation at the temperature of the experiment.

Transfer of Energy from Solvent to Solute in Liquid Organic Solutions Under Ultra-violet Excitation, S. G. COHEN and A. WEINREB, *Department of Physics, The Hebrew University of Jerusalem.* The phenomenon of transfer of energy from solvent to fluorescent solute is well known in the case of certain organic solid solutions. The works of Kallman and others have shown that such a transfer also exists in organic liquid solutions, when excited by γ-rays.

In this work the phenomenon of energy transfer in liquid solutions was found to exist in the case of ultra-violet excitation, and the problem of the nature of the transfer is being studied under these conditions. The advantages of the method are:

1) The excitation process is much simpler than in the corresponding case of excitation by γ-rays.

2) By examining the fluorescence at exciting wave-lengths for which the solvent does not significantly absorb, one can estimate the quenching of the solvent molecules on the solute fluorescence by comparing different solvents; and similarly the self quenching can be estimated by measuring the fluorescence as a function of solute concentration, under similar conditions.

Results showing the dependence of the fluorescence from *p*-terphenyl solute on concentration and exciting wave lengths have been obtained, using different solvents and solvent mixtures.

It has been shown that the phenomena of energy transfer cannot be explained on the assumption that fluorescence of the solute is produced by secondary radiation of the solvent of range greater than 0.01 mm.

Methods and Instruments (Tuesday afternoon)

Synchronization of a Pendulum Clock with Signals Derived from a Quartz Crystal Clock, M. VAN MENTS, *Research Council of Israel.* Pendulum clocks, a few of which are to be found in several institutions about the country, can have an accuracy of about less than ± 0.2 seconds a day. As is known, still greater accuracy may be obtained by installing them in a thermostat and/or in vacuum. If these measures are not taken, for many purposes it is desirable to check the clocks a few times a week with time signals from more accurate clocks. Since it is impossible, however, to correct the clocks afterwards for fractions of a second, one never can rely on the clock to within a second.

However, a way has been found to synchronize these clocks with the aid of special signals derived from a quartz crystal clock.

These signals magnetize a pair of different coils that are placed in an asymmetrical arrangement under the orbit of the pendulum. As a result of the extra forces added to the pendulum, there are now altogether three effects which cause a difference between the real and nominal period of the pendulum: a) the intrinsic divergence of the clock itself (as for instance caused by temperature influences), b) a phase effect, c) an amplitude effect.

The resultant of these effects is such that the pendulum is locked within far less than 0.1 second into synchronization with the forced pulses.

A Special Tension Source for the Electrical Standard Clock in Israel, M. VAN MENTS, *Research Council of Israel.* Owing to the frequent failure of the electric supply mains in Jerusalem, special attention must be paid to the tension source for the quartz crystal standard clocks and their accessories. To run the several services, two D.C. tensions of respectively 240 and 6.5 volts are needed without the slightest interruption. A number of floated batteries take over the service if there is a failure of the electric supply mains, and in case these batteries have lost a considerable amount of their capacity, they are reloaded when the mains supply is restored. Needless to say, this device must work automatically. In addition, special safety devices had to be installed to prevent damage that might be caused by overloading or underloading the batteries by too high voltages on the load of both the high- and low-tension circuits, and by some other possible undesirable situations.

All these and other devices have to work continuously and automatically in such a way that the time-keeping of the clocks and the service-supply is never interrupted for even 0.1 second.

A Couette Viscometer for Precision Measurement in the Centipoise Range, H. EISENBERG and E. H. FREI, *Weizmann Institute of Science, Rehovot.* Solutions of polyelectrolytes, i.e. multiply-charged compounds of high molecular weight[1], show distinct anomalous flow properties down to low rates

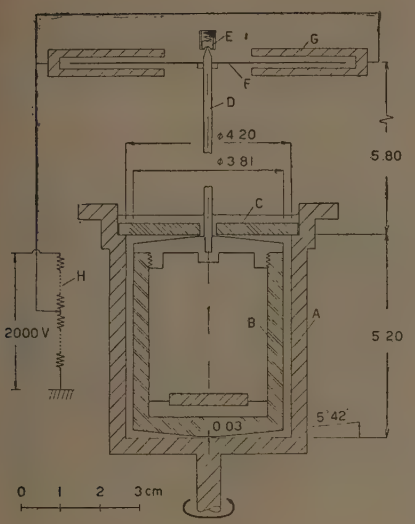

Figure 1

of shear. For the theoretical evaluation of molecular parameters and the structure of dilute solutions[2] of these substances shear stress measurements at well defined conditions are required. For this purpose, the conicylindrical type[3] instrument described below (see Figure 1) was constructed.

The inner stationary cylinder is freely floated in the sheared liquid under investigation. It is positioned by an upper pivot-and-cup bearing (outside of the liquid) and is adjusted to exert an upward thrust of about 50 mg. No other bearing is required. A stable "suspension" is obtained having a negligible frictional torque.

The restoring force is obtained by means of an electrostatic system. A variable stabilized D.C. voltage of up to 2000 volts is used. A pair of vanes, appropriately shaped, are mounted axially on the inner cylinder and move between two static pairs of plates. A knowledge of the voltage and the position of the vanes enables one to calculate the electrostatic restoring torque.

The instrument is suitable for the measurement of shearing stresses from about 2×10^{-3} dyn cm^{-2} with a precision of 0.3% at rates of shear from 0.2 sec^{-1} to 40 sec^{-1}; considerable accuracy can be attained in the determination of absolute viscosities in the centipoise range.

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2. Eisenberg, H. and Pouyet, J., 1954, *J. Polymer Scl.*, **13**, 85.
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Measurement of the Refractive Index of Light-Absorbing Substances, J. H. Jaffe, *The Weizmann Institute of Science, Rehovot.* Various methods were used for the measurement of the refractive index of substances which are highly light-absorbing, particularly interferometry. It was found experimentally possible to obtain an exact value of the refractive index only when the optical absorption was known. On the other hand, it seems to be possible to determine the absorption only when the refractive index is known, the latter being required for the correction which must be made for reflections within the sample, etc.

In every case studied experimentally, then, it became clear that it is impossible to determine the refractive index or the optical absorption independently; they must be determined together by a pair of appropriate measurements.

A New Method for Determining the Masses of Fast Charged Particles, S. Rosendorff and G. Yekutieli, *Atomic Energy Commission, P.O.B. 7056, Tel Aviv.* The mass M of a fast charged particle which does not come to rest in nuclear emulsion may be estimated by measuring the variation of ionization along its track. Let R_o be the residual range of the particle measured from the middle of its track in the emulsion. The ionization at X mm from the middle point is $I = f(R_o/M + X/m)$. ($f(R/M)$ is the ionization as a function of the residual range R). The ionization along the track is measured by counting the number of grains $G(X_i)$ (in cells of 0.5 mm, X_i mm from the middle point). The two unknowns M and R are found by applying the Principle of Least Squares to the expression

$$\Phi = \sum_i [f(R_o/M + X_i/M) - G(X_i)]^2$$

The new method was applied to a K particle of 35 mm ($\bar{G} = 1.41$ plateau value) and two pions of 7 mm ($\bar{G} = 1.53$ P.V.) and 10 mm ($\bar{G} = 1.39$ P.V.). The following mass values were obtained: 840 ± 130, 238 ± 50 and 377 ± 70 respectively.

A Transducer Using Crossed Magnetic Fields, E. H. Frei, S. Shtrikman and D. Treves, *The Weizmann Institute of Science, Rehovot.* In conventional magnetic amplifiers the flux in the magnetic material from the exciting alternating current is essentially parallel to the flux of the signal current. A system has been investigated, where the exciting magnetic field is perpendicular to the field of the signal current and of such a value as to change appreciably the permeability in the direction of the signal field (Figure 1). This leads to an alternating flux in the direction of the signal field. It consists essentially of even harmonics of the frequency of the exciting flux and its amplitude is dependent on the signal. Change of di-

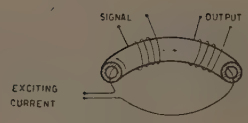

Figure 1

rection of the signal current changes the phase of this output by π.

This changing flux can produce a voltage in a winding parallel to the signal winding. It can be shown that its energy, partly or totally, derives from the exciting current.

Ferrite materials are used for the core because difficulties would arise with ferro-magnetic materials needing lamination.

Possible uses of this instrument are for a transducer for small currents[1] and for a magnetometer.

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A Clip-on Milliamperemeter for D.C. Currents, E. H. Frei and D. Treves, *The Weizmann Institute of Science, Rehovot.* Using the principle described in the previous paper[1] an instrument was built which measures direct currents by surrounding them with a magnetic circuit (without opening the electric circuit).

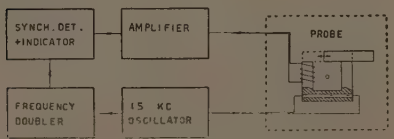

Figure 1

The magnetic transducer (probe) consists of a U-core which can be closed by a moving I-core, Figure 1. The crossed field is applied in part of the U-core, producing therein a "varying artificial airgap". The wire carrying the current to be measured is enclosed in this system. A one-tube oscillator (15 Kcs) generates the saturating current; a pick-up coil connects the probe to a two-stage amplifier. The signal is then demodulated by a synchronous detector driven by the oscillator through a doubler circuit.

Outside magnetic fields contribute a magnetic flux to the probe core. The earth's field is equivalent, in such a probe, to a current of the order of one milliampere. Shielding with Mumetal can make this influence negligible.

The demodulating output is nearly linear to the measured signal in the range of 0.1 to 4 milliamperes. The upper limit can be extended to practically any value by negative feedback. A mono-stable relay circuit makes it possible to demagnetize the probe with a decaying 50 cycle current, thereby avoiding errors due to remanence.

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A New Active Circuit Element Using the Magneto-Resistive Effect, A. Aharoni, E. H. Frei and G. Horowitz, *The Weizmann Institute of Science, Rehovot.* A bi-stable circuit, known as flip-flop is proposed, which uses no tubes.

Suppose that, in Figure 1, the resistors R_1, R_2, R_3, and R_4 are strips of bismuth. As is well known, the resistance of this metal undergoes a considerable change $R=CRH^2$, when a magnetic field H is applied. C is of the order of 10^{-8} gauss^{-2} for pure bismuth at room temperature, but is much larger at lower temperatures.

R_1, R_2, R_3, R_4 are so chosen that at a certain constant field H_o they are all equal ($=R$). The coil L produces a magnetic field KI, when the

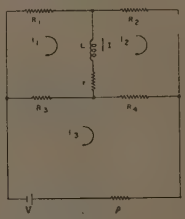

Figure 1

current through it is I. The resistors are physically placed so that R_1 and R_4 are in a field H_o+KI, while R_2 and R_3 are in a field H_o-KI. Under this arrangement any change in I changes the resistances thereby further changing I. One can calculate the current I and find that this circuit yields two stable equilibria $I=\pm I_o$ (besides the unstable state $I=0$), whenever

$$2CKH_oV/(1+CH_o^2) = r+R+\rho+r\rho/R.$$

The behaviour of such elements was verified by building an experimental model, operating at low speed.

It seems possible to use the magneto-resistive effect also in other cases in a similar way for the construction of active elements.

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Received May 13, 1954.

LETTERS TO THE EDITOR

A reaction between hypochlorite and bromate

In the course of an investigation of the oxidising action of hypochlorite-hypobromite mixtures on cellulose, the influence of bromate on the rate of decomposition of hypochlorite has been examined. It was found, surprisingly enough, that the rate of this decomposition is accelerated by bromate.

Table I shows that hypochlorite solutions of the given concentration decompose extremely slowly at pH 9. This result agrees with those of Chapin[1] and Skrabal[2].

The accelerating effect of three concentrations of bromate, viz. 0.2, 0.5 and 0.8 mole/litre respectively has been studied and is shown in Table II for 0.5 mole/litre. It cannot be explained as due to the action of a neutral electrolyte, as a parallel experiment carried out with 0.4 mole/litre of Cl^- showed only a very slow decomposition (Table I).

TABLE I

Decomposition of hypochlorite in the presence of chloride
pH=9.14 t=25 °C (NaCl)=0.4 mol/litre

Time in hours	$(\frac{1}{3}ClO_2')+(ClO')$ mol/litre $\times 10^2$	(ClO') mol/litre $\times 10^2$	$(\frac{1}{3}ClO_2')$ mol/litre $\times 10^2$
0	2.695	2.655	0.040
9 1/2	2.665	2.648	0.017
27 1/4	2.653	2.640	0.013
93	2.640	2.630	0.010
211	2.620	2.590	0.030

It is suggested that a slow reaction between hypochlorite and bromate takes place according to the following overall equation:

$$HBrO_3 + HOCl = HClO_3 + HOBr$$

The equilibrium constant and free energy change for this reaction, as calculated from the values of the free energy of formation of the reactants and the products[3], are 218 and −3.2 Kcal respectively, which proves that the reaction is thermodynamically feasible.

The formation of chlorate might possibly proceed by a number of intermediate steps:

$$HBrO_3 + HOCl = HClO_2 + HBrO_2 \quad (I)$$
$$HBrO_2 + HOCl = HClO_2 + HOBr \quad (II)$$
$$HClO_2 + HOCl = HClO_3 + HCl \quad (III)$$

The initial increase of available oxidising capacity as determined by the arsenite-iodine method (see Table II and Figure 1) would indicate reaction (I). As chlorite ions are not determined by the arsenite-iodine reaction, this increase would be due to the formation of bromite. (Unfortunately the latter cannot be determined directly).

TABLE II

Decomposition of hypochlorite in the presence of bromate
pH=9.15 t=25 °C $(KBrO_3)$=0.5 mol/litre

Time in hours	$(ClO')+(BrO')$ mol/litre $\times 10^2$	(ClO') mol/litre $\times 10^2$	(BrO') mol/litre $\times 10^2$
0	2.664	2.618	0.046
1	2.720	2.628	0.092
2 1/2	2.740	2.640	0.100
5 1/2	2.683	2.602	0.081
17	2.453	2.410	0.043
40 3/4	2.262	2.228	0.034
71 1/5	2.108	2.084	0.024
114 1/2	2.014	1.986	0.028
191	1.990	1.952	0.038
311	1.959	1.930	0.029

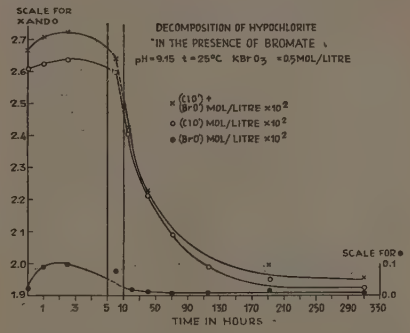

Figure 1

As at the same time some hypochlorite will react with chlorite to form chlorate according to reaction (III), which is known to proceed at a rapid rate[1] and thus cause the oxidising capacity to decrease, the concentration of the bromite would actually be even higher than indicated by the maximum of the curve. Reaction (II) would account for the small amount of hypobromite found in the reaction mixture. The hypobromite reacts relatively rapidly with hypochlorite to form bromate and chlorate[4], which explains the low concentration of hypobromite found.

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Received November 6, 1953.

Copolymerization of α-amino acids with ω-aminohendecanoic acid

Poly-ω-aminohendecanoic acid[1] is a polyamide ("Rilsan") used in the production of fibres and plastics. Owing to the absence of reactive side groups, it cannot be dyed by the usual techniques for natural fibres. It is therefore to be expected that the dyeability and absorptive capacity of polyamides might be improved by the introduction of polar and hydrophilic groups into the polymeric chain. A copolymer of the required type was prepared several years ago[2] by the heating of ε,*N*-carbobenzoxy-α,*N*-carboxylysine anhydride with ω-aminohendecanoic acid[1] and by the subsequent removal of the carbobenzoxy groups.

A copolymer containing tyrosine and ω-aminohendecanoic acid was prepared by means of a similar reaction, copolymerization of the *N*-carboxy-α-amino acid anhydride with ω-aminohendecanoic acid. Tyrosine monomer was chosen for copolymerization since it was expected to facilitate dyeing by adsorption and by direct coupling with aromatic diazonium salts. In addition, the polytyrosine[3] formed as a byproduct can readily be removed, and the tyrosine in the copolymer can easily be determined.

The copolymer of tyrosine and ω-aminohendecanoic acid was prepared by mixing *O*-carbobenzoxy-*N*-carboxy-*L*-tyrosine anhydride[3] (5 mols) with ω-aminohendecanoic acid[1] (95 mols), heating to 200° in vacuo for eight hours, cooling, and removal of the carbobenzoxy groups of the copolymer obtained by a 33% solution of hydrogen bromide in glacial acetic acid[4]. The decarbobenzoxylated copolymer was precipitated with absolute ether, washed with ether and ethanol, and dried in vacuo. The product obtained is soluble in hot dimethylformamide. It was purified from traces of tyrosine and polytyrosine by repeatedly dissolving in dimethylformamide and precipitating alternately with 2N sodium hydroxide and 2N hydrochloric acid. The copolymer was dried in vacuo over sulphuric acid and potassium hydroxide. A sample of the copolymer (100 mg) was hydrolysed for 72 hours in conc. hydrochloric acid (2 ml) at 120°. The dried hydrolysate was dissolved in 0.1 N sodium hydroxide and the amount of tyrosine determined spectrophotometrically at 2935 A[3,5]. It was found that the copolymer contains 3.3 tyrosine residue-mols per each 100 residue-mols of the copolymer. Coupling of the copolymer with various aromatic diazonium salts (e.g. *p*-nitroaniline, sulphanilic acid, *p*-bromoaniline) yielded polymeric azo dyes.

It could further be demonstrated that even free α-amino acids, such as tyrosine, phenylalanine, leucine and valine, can be copolymerized with ω-aminohendecanoic acid. All copolymerizations of ω-aminohendecanoic acid with varying amounts of free α-amino acids were carried out as previously described. The copolymers obtained were soluble in hot dimethylformamide and were purified by repeatedly dissolving in dimethylformamide and precipitation with hot water. Hydrolysis of the dried copolymers was carried out after each purification as previously described, and the amounts of the different α-amino acids were determined by the ninhydrin-carbon dioxide method[6], or, in the case of tyrosine, spectrophotometrically (pH 13, 2935 A)[3,5].

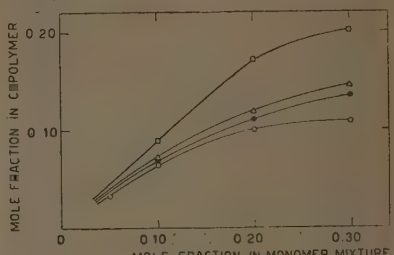

Figure 1

Mole fractions of the α-amino acids in copolymers with ω-aminohendecanoic acid, as a function of the mole fractions of the α-amino acids in the monomer mixture.
○ tyrosine; ● leucine; △ valine; □ phenylalanine

Figure 1 shows the mole fractions of α-amino acids in the hydrolysates of the various copolymers obtained on copolymerization of different initial mole fractions of both components.

The copolymers were purified by the precipitation method until the mole fraction of the α-amino acid reached a constant value, indicating that in each case a true copolymer was found.

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Received January 25, 1954.

A simple semi-solid tryptone-urea medium as aid in the diagnosis of Enterobacteriaceae

Many media have been produced in the past, which give simultaneously various characteristics of the intestinal group of organisms, and facilitate their routine diagnosis.

The Bacto Sim medium was devised as an aid in the identification of members of the *Salmonella* and *Shigella* groups, and the one-tube medium enables the detection of H_2S production, indol, production and motility.

Friewer and Shaughnessy[1] described a lead semi-solid medium, as a preliminary screening medium in the classification of the *Salmonella-Shigella* group.

Many modifications of those media are in use, and in all cases urease activity, in some indole production, are tested separately.

The one-tube medium described below facilitates the detection of a) urease activity, b) indole production and c) motility, and gives together with either, a double sugar iron medium (Kligler's), or a triple sugar iron medium (Difco), or a T.S.I. Agar (BBL), — the general characteristics of the organism, belonging to the *Enterobacteriaceae*. The routine diagnosis is reduced to the inoculation of the two test tubes — a procedure which saves time and serum in the final slide agglutination test.

Preparation of the semi-solid medium

Bacto-tryptone (Difco)	10 g
Phenol-red 0.1%	5 ml
Aqua dest.	1000 ml

50 or 100 ml portions of the medium were distributed into bottles and sterilised in the autoclave. This stock solution keeps indefinitely. Twenty-four hours before use 0.5% urea from a 50% conc. solution (sterile) were added. The proper amount of melted 2% nutrient agar was added to make a final agar conc. of 0.3—0.4% (8—10 ml to 50 ml medium). The latter was distributed into small sterile test tubes (10×75 mm), put into ice-box for 30 min. for semi-solidification, then transferred into the incubator to test for sterility. The medium was kept at room temperature until used.

Mode of inoculation of the medium

Using a straight platinum wire, the semi-solid medium was punctured to a depth of not more than 1—1.5 cm (1/3 of the total depth of medium in test tube). For the indole test, a strip of hard filter paper, previously soaked with the modified Ehrlich-Pringsheim reagent[2]* and then allowed to dry, was inserted between the cotton plug and mouth of the test tube.

Indole production was detected by a reddish-violet colour appearing on the yellow paper strip. Urease activity was indicated by the alkaline reaction as shown by the Phenol-red indicator. Non-motile organisms grew along the line of inoculation, with a small conus of growth near the surface of the medium at the place of puncture.

Motile organisms gave a swarming growth throughout the medium, totally covering the surface of the medium.

* Modified Ehrlich-Pringsheim reagent: *p*-dimethylamidobenzaldehyde 5.0 g, methyl alcohol 50.0 ml, 66% conc. phosphoric acid 10.0 ml. A comparative study of the indole test by the paper strip method and by the Kovacs reagent method yielded the same results. The paper strip method is more convenient as it saves time.

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Failure to detect complement fixing antibodies in the serum of mice immunized against homologous tumors with the aid of adjuvants*

The problem of the existence of distinct tumor antigens is still highly controversial and the considerable amount of research devoted to it so far has been mostly disappointing. A comprehensive review on the subject has been recently published[1].

In spite of the numerous failures met with in this field, it was considered that the problem is important enough to warrant further research as long as all lines of approach have not been completely exhausted. The advent of Freund's adjuvant technique[2] and the striking successes obtained with it in evoking immune responses to weak antigens, particularly in work on allergic encephalitis where tissue antigens are involved[3], prompted the initiation of a similar experiment on tumor immunity.

Materials and methods

Animals. C57 black mice were used throughout this experiment.

Tumor. 2 mg methylcholanthrene in 0.5 cc olive oil were inoculated subcutaneously to 5 mice. 3 months later a tumor was obtained in the only surviving mouse and it proved on histological examination to be a polymorphocellular sarcoma. The tumor was subsequently maintained by transplanting it at 3 weeks intervals.

Preparation of antigen for immunization. When the tumor had become well established (about the 4th passage) antigen for immunization was prepared in the following manner: — Mice were killed with ether and the 3 weeks old tumors (appr. 1 cm in diameter) were aseptically removed, ground with sterile glass sand in porcelain mortars and a 20% suspension prepared in buffered saline (pH 7.2). The suspension was left overnight in refrigerator (if used immediately it caused formation of tumors) and then centrifuged for 15 minutes in the cold room (+4°) at low speed. The supernatant was emulsified with Bayol F and Arlacel A according to the method of Salk and Laurent[4]. Autoclaved and dried TB bacilli were also added to the oil phase to a concentration of 0.5 mg per 0.1 cc of antigen-adjuvant mixture.

Preparation of the immune sera. Immune sera were prepared in the homologous C57 black mice, thus avoiding iso-immune reactions. 3 inoculations of antigen-adjuvant mixture were administered at monthly intervals, each inoculum consisting of 0.1 cc intramuscularly into the thigh of the left hind foot and 0.1 cc subcutaneously. 1 month after the last inoculation the animals were bled from the heart under ether anesthesia and the pooled serum (obtained from 18 mice) was stored in the deep freezer (—25°) without preservative.

* Supported by the Jacob I. Schaffer Memorial Research Scholarship in Medicine.

Complement fixation tests. The tests were performed following the sensitive technique of Thornton et al[5], according to which complement is titrated in the presence of antigen-antiserum mixtures. The tumor suspension used for immunization served also as antigen for the complement fixation test, but diluted 1/2, this being the minimal dilution which did not exhibit anticomplementary action at higher concentrations. Serum dilution was $1/20$, as normal sera had some anticomplementary action at higher concentrations. The hemolytic system was a 2% Sheep Red Blood Cells suspension sensitized with an equal volume of 3 MHD anti-sheep hemolysin.

RESULTS AND COMMENTS

Results of a typical experiment are presented in Table I.

TABLE I

Complement fixation test with "antitumor" and normal mouse sera

Complement titrated in presence of	Complement titres (Dilutions of complement at which 50% fixation was obtained)
Antitumor serum+antigen	1/80
" " +saline	1/90
Normal mouse serum+ antigen	1/100
Normal mouse serum+ saline	1/100
Antigen+saline	1/100

The degree of fixation obtained is too small to be considered significant.

The complement fixation test was repeated 3 times with practically identical results.

Whilst it is obvious that this is only a preliminary experiment which by no means could be intended or expected to settle definitely the question of the existence of distinct tumor antigens, it is however believed that further research with the adjuvant technique, employing different tumors and different adjuvants, could bring weighty evidence for or against the theory of homologous tumor antigenicity.

Acknowledgements. Thanks are due to the Department of Experimental Pathology and Cancer Research, Hebrew University-Hadassah Medical School for providing the mice used in this study and for much help in various ways.

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Received January 1, 1954.

Sedimetric method for potassium determination

In our development work and for following up the progress in the factory of potash production, we were constantly hampered by not having a quick and relatively accurate method for potassium determination. The disadvantage of excessive time consumption is common to the numerous existing and proposed methods. Even the accepted standard method, the perchlorate method which gives highly accurate and reproducible results, suffers from the time disadvantage and the necessity for the prior separation of SO_4^{--} ions. There is a pressing need then for a reagent which can be used to estimate potassium rapidly and accurately[1].

M. Lavoye[2] sugested a gravimetric method for potassium determination which depends upon the small solubility of potash alum in concentrated solutions of aluminium sulphate containing sulphuric acid. The authors filtered off and weighed the precipitated alum, and the results seem to be accurate and reliable. However, this method is also very time consuming as it necessitates a waiting period of 12 hours prior to the filtration of the alum.

We have developed a method based on the low solubility of alum in concentrated solutions of aluminium sulphate. It has been found that under certain conditions, the volume of precipitated alum is directly proportional to the original amount of potash present; hence, determination of the volume of the precipitated alum obviates the necessity for filtering and weighing.

Preliminary investigations

To determine the practicability of this method in industrial determinations, investigations were made on the influence of temperature and of salts generally found to occur concurrently with the potassium on the final equilibrium between precipitated K^+ and that remaining in solution, To this end, various solutions of KCl were prepared containing NaCl or NaCl and $MgCl_2$ in proportions corresponding to those in samples of crude potash material. The potassium was precipitated with aluminium sulphate solution (Sp. gr. 1.345, 25°C), the amount of KCl solution. being varied in its relation to the aluminium sulphate corresponding to the relations applied in the final analytical method. The equilibrium was investigated at 25°C and at 35°C. The potassium content of the supernatant liquid was determined after equilibrium had been established. In all cases, the potassium content of the supernatant liquid was determined by the cobaltinitrite method, as the high SO_4^{--} content of the system made the perchlorate method too troublesome. The results are recorded in Table I.

TABLE I

Sp. gr. of $Al_2(SO_4)_3$ *solution used for precipitation* = 1.345 (25 °C)

KCl solution containing g/litre			ratio of volumes of $Al_2(SO_4)_3$ solution to KCl solution	% K^+ of total K^+ precipitated at	
KCl	NaCl	$MgCl_2$		25 °C	35 °C
125	125	—	15:8.0	91.7	86.2
125	125	—	15:4.0	94.2	89.0
92	64	138	15:7.5	93.2	90.0
92	64	138	15:4.0	93.5	87.0

Graphs showing the relationship of volume of precipitated alum to amount of potassium originally present in the solution were plotted for varying volumes of solutions and varying temperatures. Iso-thermals at 25°C, 35°C and 40°C for volumes corresponding 4 ml and 8 ml KCl-solutions with 15 ml aluminium sulphate solution (sp. gr. 1.345)* were prepared, thus facilitating the analysis of potassium under various conditions and obviating the necessity of working under fixed temperatures (Figure 1). In the final analysis the temperature must be observed only during precipitation. The temperature during centrifugation is irrelevant since the chief part of solution which is in equilibrium with the precipitate is withdrawn.

Preparation of graphs

Potassium chloride A. R., dried at 130 °C, was used to make up the following solutions:

TABLE II

100 ml solution containing		
KCl (g)	NaCl (g)	KCl (%)
25.00	—	100
18.75	6.25	75
12.50	12.50	50
6.25	18.75	25
2.50	22.50	10
1.25	23.75	5

4 ml of each solution were introduced into specially designed cylindrical centrifugal tubes (length 110 mm, dia. 14 mm), graduated at intervals of 0.2 ml. Into the top of this tube was inserted, by means of a standard ground joint, a funnel-like 80 mm long tube, the diameter of which above the joint widened to 25 mm (Figure 3). 15 ml of the aluminium sulphate solution were introduced into this apparatus and the mixture immediately stirred by means of a thin glass rod flattened at the lower end. The aluminium sulphate reagent was prepared by dissolving the equivalent of 700 g hydrated $Al_2(SO_4)_3$ in 1000 ml hot distilled water. After cooling and filtering, the specific gravity of the solution was brought to 1.435 (25 °C). The precipitated alum was allowed to settle for about ten minutes until it had sunk approximately 2—3 cm below the joint. The clear supernatant liquid

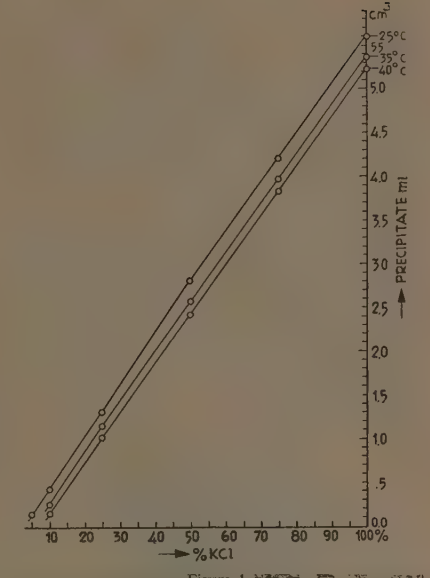

Figure 1

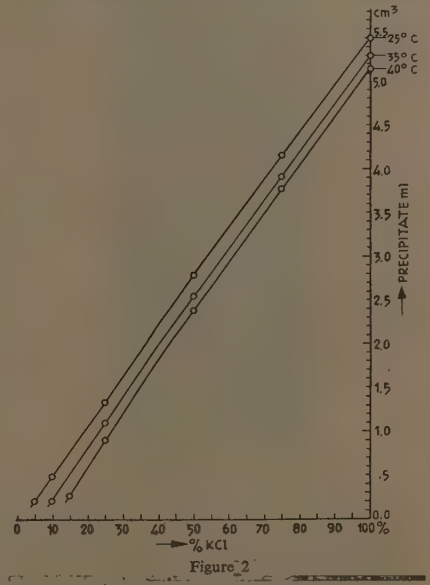

Figure 2

was then withdrawn with a pipette, the upper funnel-like part removed and the lower tube placed in the centrifuge. This latter should not

* If the work is carried at a lower temperature, the sp. gr. should be lowered to 1.330.

have an angle-head, as horizontal sedimentation is required to obtain correct readings of volume of precipitate. The solution is centrifuged for 10 minutes at a rate of 3000 r.p.m. after which the tube is taken out and the volume noted.

It was found that the various volumes of precipitate were in all cases directly proportional to the varying amounts of KCl, irrespective of the amount of added NaCl; hence the alum volume — % KCl graph was represented by almost straight lines.

Similar graphs for determinations using 8 ml of solution — in use for samples poor in KCl — were prepared (Figure 2). The standard KCl solutions from which those graphs were prepared contained, in addition to NaCl, also $MgCl_2$. Here also it was found that those added salts had no practical effect on the final volume of potash alum and that the alum volume — % KCl graph was represented by almost straight lines.

Figure 3

Procedure for practical determination

A given quantity of the salt to be analyzed, the amount depending on the nature of the material and its potash content, is dissolved in a 100 ml volumetric flask (see Table III which gives the amounts of the various materials to be taken for analysis). If the sample is suspected to contain calcium, 10 ml of 20% Na_2SO_4 solution are added. Water is added to the mark and the solution thoroughly shaken. It is then filtered from insoluble matter and an aliquot portion withdrawn by means of a pipette into the graduated centrifuge tube. The procedure then follows as outlined above for the preparation of the diagram.

After centrifugation and after reading the alum volume, the percentage of KCl is drawn from the corresponding graph.

TABLE III

Material	Amount	Dissolved in ml H_2O	Aliquot portion withdrawn	Amount in aliquot portion
Sylvinite (KCl+NaCl) containing 10—100% KCl	25 g	100	4.0 ml	1 g
Sylvinite containing less than 10% KCl	25 g	100	8.0 ml	2 g
Carnallite containing ~20% KCl (+NaCl+$MgCl_2$)	40 g	100	7.5 ml	3 g
Brine with more than 40 g KCl/litre	4 ml	—	—	4 ml
Brine with less than 40 g KCl/litre (rich in $MgCl_2$)	75 ml	100	8 ml	6 ml

Table IV enables a comparison to be made between the results obtained by the alum method and those by the perchlorate one.

TABLE IV

Material analyzed	Alum method % KCl	Perchlorate method % KCl
Carnallite	19.5	19.55
"	13.5	13.23
"	18.0	18.05
"	19.8	19.50
"	20.7	20.30
"	15.5	15.60
"	22.4	22.70
"	21.2	21.30
Carnallite + Sylvinite	27.8	27.92
"	30.2	30.30
"	25.3	25.43
Sylvinite	54.0	54.18
"	45.7	45.92
"	39.0	39.08
"	61.5	61.54
"	63.5	63.76
"	33.7	33.69
"	90.6	90.87
"	83.5	83.76
"	89.2	89.28
"	97.6	98.56
"	9.6	9.90
"	6.2	6.25
Factory-brine ...	34.3 g/l	33.1 g/l
"	63,3 "	63.9 "
"	97.0 "	97.3 "

The variations are within the limits of error permitted for industrial purposes.

Summary and Conclusion

A sedimetric method is described which permits the rapid and comparatively accurate determination of potassium. The potassium is precipitated as potash alum and the volume of the precipitated alum, which is a function of the amount of the potassium originally present, is measured after centrifugation.

Acknowledgment

The authors wish to thank Dr. M. R. Bloch, Director of Research and Development of the Dead Sea Works Ltd., Jerusalem, for his help and useful suggestions.

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Amplification of microwaves of 1 cm by means of a cold cathode D. C. discharge

In the course of measurements of the attenuation of microwaves by the various regions in a D.C. discharge, we have observed negative attenuation or amplification of microwaves at a frequency of 24000 Mc. In view of the importance of these results, both in explanation of propagation of radio waves from the sun as well as possible practical applications, we report these preliminary results.

The discharge tube, 50 cm long and with two aluminium electrodes of 5 cm diameter, contained helium at 1-3 mm of pressure. The low power microwaves from a klystron tube were beamed at right angles to the discharge tube, with the electric vector parallel to the axis of the tube. The transmitted radiation was detected by means of a crystal detector. The attenuation was measured by means of a calibrated attenuator. Considerable difficulties were experienced with interference from the walls of the discharge tube.

At a distance of 15 mm from the cathode near the beginning of the negative glow, we observed amplification or negative attenuation in a very narrow region of the order of 1 mm. A secondary dip of attenuation but no amplification was observed at 31 mm from the cathode. The amplification increases with the increase of the current in the discharge tube. There is some indication that at stronger currents the region of amplification moves slightly towards the cathode.

Merrill and Webb[1] and recently other[2,3] authors have shown that oscillations occur in a narrow layer close to the cathode, in about the same region we have observed the microwave amplification. Unlike our experiments, these oscillations have been observed in a hot cathode discharge and at frequencies not larger than 10^3 Mc[4].

Tonks and Langmuir[5] have shown that electrons in an ionized gas had a natural frequency w_f of oscillation given by $(4\pi Ne^2/m)^{\frac{1}{2}}$ where N is the electron density, e and m the charge and mass of the electron. Bohm and Gross[6] have amplified the theory and shown various conditions for the excitation of these oscillations. These plasma oscillations. however, have no group velocity and, therefore, energy cannot be propagated outside the oscillating region. Even if oscillations were induced by the microwave frequency, it is unlikely that much of the energy could be transmitted through the plasma. The region of plasma oscillation is near the region where strong attenuation occurs due to the fact that the refractive index tends to zero (for reasonable values of the electron collision frequency).

The oscillation detected by Merrill and Webb as well as by others is usually explained by velocity modulation of the primary electrons from the hot cathode by means of the plasma oscillations. This causes electron bunching and transmission of the energy. Our results could only with difficulty be explained by means of a velocity modulation of the electrons near the negative glow by the incident microwave frequency.

Bailey[7] has shown that theoretically amplification of electromagnetic energy can occur at certain frequency bands, lower than the plasma oscillation frequency, either if the ionized gas is pervaded by static electric and magnetic fields, or if the waves are propagated obliquely to the drift velocity of the electrons in the absence of a magnetic field. In our experiments the propagation of the microwaves was at right angles to the velocity of the electrons, and no external magnetic field was present.

From our results one can conclude that there is a narrow region near or in the negative glow in which microwaves are amplified. It seems likely that this is the same region where violent oscillatory potentials have been observed in a hot cathode discharge tube.

Experiments are in progress in which the amplification is being measured as a function of pressure, microwave frequency, polarization, and under external fields. Detailed results of these measurements, as well as the application of measurements of attenuation to determine electron densities and electron collision frequencies, will be reported later.

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NEWS AND VIEWS

SCIENTIFIC CONVOCATION IN HONOUR OF PROFESSOR EINSTEIN'S 75th BIRTHDAY

On Sunday March 14, the seventy-fifth birthday of Albert Einstein was celebrated at a scientific convocation held in Jerusalem under the joint auspices of the Hebrew University of Jerusalem, the Technion, Israel Institute of Technology, Haifa, The Weizmann Institute of Science, Rehovot, and the Research Council of Israel.

Highest tribute was paid to Professor Einstein by the President of Israel, Mr. Y. Ben-Zvi, by the Prime Minister of Israel, Mr. M. Sharett, and by President of the Hebrew University, Professor B. Mazar.

Various aspects of Professor Einstein's scientific activities were discussed. Professor G. Racah of the Hebrew University spoke on the quantum theory, Prof. Ch. Pekeris of the Weizmann Institute of Science on Brownian movement, Professor N. Rosen of the Technion on the teachings of relativity, and Professor S. Sambursky, Director of the Research Council, on the cosmology of Einstein.

In a cable addressed to the Convocation, Professor Einstein conveyed his "deep appreciation of this honour" and expressed his "best wishes for the continued success of the scientific and cultural endeavours of the institutions of higher learning in Israel."

THE FIFTH INTERNATIONAL COLLOQUIUM FOR SPECTROSCOPY

The above Colloquium will take place at Gmunden (Salzkammergut, Austria), August 30 — September 3, 1954.

The section of absorption and emission spectroscopy will be kept up as previously. It is suggested that the application of molecular spectroscopy to the examination of technical artificial fibres be dealt with. It is recommended that in the field of emission spectroscopy, analytics of non-conductors and the alloys of copper be chiefly discussed.

In addition to the lectures, discussion evenings will be arranged at which one or more reports will be given of personal experiences and there will be free discussions.

Organizations in various countries are asked to submit short summaries of no more than five lectures (15 minutes each) before May 30, 1954.

Comments are requested, and may be sent to the above address.

9th INTERNATIONAL CONGRESS OF GENETICS

A resolution passed at the Second Plenary Session of the 9th International Congress of Genetics held at Bellagio, August 30, 1953:

"The Congress asks the International Committee not to recommend that the next Congress be held in any country to which it may be expected that scientists would be refused permission to enter on grounds of race, nationality, religion, place of birth, or political associations past or present".

It was decided afterwards that the next Congress be held in 1957 or 1958 in Montreal, Canada.

CORRIGENDA (to Vol. III. No. 3)

p. 240 Table V *should read:* (See attachment)

p. 281 Summary of E. Shmueli, l. 7: *for* significant by lower when *read* significant by higher when

FRAENKEL, A. H.

The Intuitionistic Revolution in Mathematics and Logic, Bull. Res. Counc. of Israel, 1954, 3, 283—289.
A description is given of the recent attitude about the essence and the foundations of mathematics, called neo-intuitionism by its initiator Brouwer. Among its features are: identification of mathematical existence with construction; rejection of the logical principle of the excluded middle; abandonment of mathematical continuity in its classical sense. For the physicist, acceptance of this attitude must depend on the invention of alternative methods in mathematical analysis.

RACAH, G.

The Competition between the Low Configurations in the Spectra of the Iron Group, Bull. Res. Counc. of Israel, 1954, 3, 290—298.
Interpolation formulas are given for calculating the relative positions of the terms with highest multiplicity of the different low configurations in the spectra of the iron group. Applications to other configurations and to other rows of the periodic table are also considered.

TEITELBAUM, P.

Magnetic Moments of Odd Nuclei in jj Coupling, Bull. Res. Counc. of Israel, 1954, 3, 299—303.
Magnetic moments of odd nuclei containing n equivalent particles outside the core of completed shells treated on the assumption that each particle outside the core possesses a magnetic moment equal to the one particle value in the Schmidt diagram, but combining all such particles, both protons and neutrons, into states classified by the isotopic spin T, resultant angular momentum J, reduced seniority s and reduced isotopic spin t. Making use of fractional-parentage-coefficients the general formula of magnetic moments for odd nuclei in states $J=j$, $s=1$ and smallest T is obtained. It is seen that generally the magnetic moments should deviate from the Schmidt lines and fall between the lines.

FOX, D.

Production of Antiprotons in P-P Collisions, Bull. Res. Counc. of Israel, 1954, 3, 304—307.
The cross section for the production of a proton-antiproton pair by the collision of two protons is calculated in the neighbourhood of threshold, using the Feynmann method and assuming pseudoscalar mesons with pseudovector coupling. It is shown that certain symmetry arguments may be used, without performing the calculations, to obtain some information about the angular and energy dependence of the process; these arguments are also applicable to other weak coupling theories than that considered in this paper. The results are compared with those obtained by Fermi and by Taketani and Machida.

NEUMANN, J.

On the Second Harmonic of the Annual Variation of Solar Radiation and the Consequence of its Zeros, Bull. Res. Counc. of Israel, 1954, 3, 308—311.
A relatively simple form is obtained for the Fourier coefficient of the semi-annual wave of solar radiation reaching the outer atmosphere. The assumptions involved in the derivations are of a light nature. The latitudinal variation of this wave is studied in brief and it is shown, amongst other things, that the amplitude vanishes for the middle latitudes of each of the two hemispheres.

Sverdrup has found that the annual variation of evaporation from the oceans in middle latitudes is a double wave. Properties of the latitudinal variation of the semi-annual wave of solar radiation and features of conduction of heat in water bodies are used to explain Sverdrup's results.

COHEN, S. and SHMORAK, M.

An Attempt to Detect the Electrolytic Migration of Radioactive Interstitial Silver Ions Produced in a Silver Chloride Crystal by Slow Neutron Irradiation, Bull. Res. Counc. of Israel, 1954, 3, 312—315.

The irradiation of ionic crystals with thermal neutrons should lead to the production of radioactive ions which are displaced from their normal lattice sites by nuclear recoil in a process similar to that occurring in the Szilard-Chalmers effect. Under certain conditions, one might be able to detect the the electrolytic migration of these interstitial radioactive ions, using radioactive counting techniques. From the negative results in the case of silver chloride crystals, one concludes that at —80 °C the mobility of interstitial silver ions is smaller than 7×10^{-8} cm^2 sec^{-1} volt^{-1}, and that the activation energy for diffusion in greater than 0.23 e.v.

JAFFE, A. A.

The M X-Rays from Radium D and the M X-Ray Fluorescence Yield of Bismuth, Bull. Res. Counc. of Israel, 1954, 3, 316—320.

An investigation of the soft electromagnetic spectrum of RaD down to an energy of 1 kev, by means of a specially designed proportional counter, shows that *M* x-rays are emitted. The intensity of *M* x-rays from such a source has been measured relative to that of *L* x-rays from the same source. Consideration of the decay scheme of RaD, and of the processes involved subsequently to *L* ionisation of RaE, show that such a measurement enables an estimate of the *M* fluorescent yield of bismuth to be made. The total *M* yield of RaE (bismuth) is thus found to be 0.037 ± 0.007.

MUHSAM, H.

A Probability Approach to Ties in Rank Correlation, Bull. Res. Counc. of Israel, 1954, 3, 321—327.

The distributions of various rank correlation coefficients of all pairs of rankings containing ties which can be obtained from a pair of untied rankings by tying members in all different possible ways is studied for a particular numerical example. These distributions are shown to be skew and the respective arithmetic means to differ from the coefficient of correlation of the pair of untied rankings.

ROSEN, N.

Some Cylindrical Gravitational Waves, Bull. Res. Counc. of Israel, 1954, 3, 328—332.

Several solutions of the gravitational field equations of the general theory of relativity are obtained, representing cylindrical waves. It is found that it is possible to set up solutions representing such waves which are free from singularities.

FEKETE, M.

On the Semi-Continuity of the Transfinite Diameter, Bull. Res. Counc. of Israel, 1954, 3, 333—336.

The transfinite diameter $\tau(s)$ of an infinite compact set s of points in the complex z-plane is a set-function whose properties have been much investigated since its first introduction, some three decades ago, by the author. The present note is a new contribution to the study of variation of $\tau(s)$ when s is subject to a continuous variation.

The fact that the transfinite diameter of the ρ-neighbourhood $s(\rho)$ of an arbitrary set s tends to $\tau(s)$ as $\rho \to 0$ is a simple consequence of the original definitions[4]. The counterpart of this semi-continuity (from above), a phenomenon which is more difficult to recognize, is the objective we now pursue. Some special categories of compact sets s, described in detail in the sequel (cf. Theorem I and II), will be considered with the aim of showing the possibility of their approximation (from within) by lemniscates L such that the ratio $\tau(s)/\tau(L)$ is as near to 1 as we like.

LOW, W.

Some Implications of a Smaller Nuclear Radius, Bull. Res. Counc. of Israel, 1954, 3, 337—347.

The nuclear radius is a very sensitive parameter in calculations of nuclear quadrupole moments, gamma ray transitions, excited states of even-even nuclei, and isotope shifts. It is shown that a reduced radius of about $R = 1.2 \times 10^{-13} A^{1/3}$ cm, as found by Fitch and Rainwater, gives much better agreement between experimental and calculated values. The implications of a non-uniform charge distribution are also explored.

OLLENDORFF, F.

On the Radiation Induced Acceleration of Elementary Electric Particles, Bull. Res. Counc. of Israel, 1954, 3, 348—358.

The technically highly developed apparatus for the acceleration of electrically charged elementary particles known up to the present are based upon the application of longitudinal, electromagnetic waves, whose phase-velocity is synchronized with the corpuscle velocity of the particles. The application of the radiation pressure of transversal, electromagnetic waves gives rise to the possibility of a new type of accelerator using an asynchronous driving mechanism. The mathematical treatment of this effect requires the introduction of a force of the radiation reaction into the equations of motion. Integration of these equations is carried out for the case of the particle at rest and for the case of the particle in motion. The radiation pressure is always so small under terrestrial conditions that it is not suitable for technical applications; but it can transfer considerable energy to a particle through its action along astronomical paths. However, if a stationary, magnetic field, which is directed parallel to the radiation, is superposed on the electrodynamic working field, then resonance of the thus excited circular movement with the frequency of the progressing wave can enhance the radiation pressure by several orders of magnitude. Choosing the frequency suitably this resonance is produced automatically by the relativistic Doppler effect on the moving particle. It appears in the light of these results that it would be possible to build an asynchronously driven radiation cyclotron, which could enlarge the scope of the existing synchronous ion accelerators. Further studies of these phenomena give insightin to the physics of cosmic rays. The possibility of carriers in the cosmic rays is assumed, which originate in the strongly magnetized stellar spheres of the nature of sunspots. It has to be noted however that only blurred resonance phenomena can be expected there due to disperse electromagnetic waves; a quantitative formulation of this case requires a statistical treatment not given in this paper.

DE SHALIT, A.

Angular Momentum in Non-Spherical Fields, Bull. Res. Council of Israel, 3, 359—363.

Bohr's formula for the angular momentum operator of a quantized liquid drop is derived for a general non-spherical field. It is shown that the commutation relations of the field variables and the structure of the angular momentum operator result from the required invariance of the interteraction under rotations of space.

DVORETZKY, A., ERDOS, P. and KAKUTANI, S.

Multiple Points of Paths of Brownian Motion in the Plane, Bull. Res. Counc. of Israel, 1954, 3, 364—371.

The paper is concerned with paths of (mathematical) Brownian motion in the plane and studies the existence of k-multiple points, i.e. of points through which the path passes k times. The main result is that almost all paths, i.e. with the exception of a set of paths having probability zero, contain multiple points of arbitrarily high finite multiplicity.

REINER, M.

Second Order Effects in Elasticity and Hydrodynamics, Bull. Res. Counc. of Israel, 1954, 3,372—379.

Second order effects appear in quasi-linear elasticity when the strain is finite. They reveal themselves through the requirement that simple shear can be maintained only in the presence of (i) an isotropic tension or pressure (Kelvin-effect). (ii) cross-stresses consisting of a tensile stress in the direction of the displacement and a compressive stress in the direction of its gradient (Poynting effect). These two effects are illustrated on the example of simple torsion. The cross-elasticity term in the general law of isotropic elasticity permits the appearance of cross-stresses of *any* sign. No second-order effects of this kind exist in quasi-linear viscosity. The cross-viscosity term in the general law of viscosity reveals the presence of cross-stresses of equal sign. This is illustrated on the example of the centripetal-pump phenomenon.

LEVITZKI, J.

Some Theorems Concerning Associative Zorn Rings, Bull. Res. Counc. of Israel, 1954, 3, 380—384.

A ring S is called an associative Zorn Ring or in short an I-ring, if S is associative and if every non-nil right ideal in S contains a nonzero idempotent. A ring whose homomorphic images are I-rings is called a faithful I-ring (in short: an FI-ring). Any algebraic algebra is an FI-ring. In a recent paper (compare paper, ref. 13) the author has studied the structure theory of certain rather general types of I-rings and FI-rings. In the present note, some of the results of that paper are applied in order to derive certain theorems which hitherto were known to be valid only for finite algebras and for rings with a minimum condition for right ideals.

AGMON, S.

On the Singularities of a Class of Dirichlet Series, Bull. Res. Counc. of Israel, 1954, 3, 385–389.

This paper deals with some properties of the singularities of functions $f(s)$ represented by Dirichlet series with exponents $\{\lambda_n\}$ satisfying $\liminf(\lambda_{n+1}-\lambda_n)>0$. Assuming the imaginary axis to be the axis of convergence, it is shown that if the only singularities of $f(s)$ in a segment I of the imaginary axis whose length depends on $\{\lambda_n\}$ are the simple poles $i\alpha_q\ (q=1, \ldots, k)$, then the character of all other singularities on the imaginary axis is determined in a very specific way. In particular it is shown that if $i\alpha$ is a singularity of $f(s)$, isolated on the imaginary axis, then it is necessarily a simple pole and α is a linear combination with integral coefficients of $\alpha_1, \ldots, \alpha_k$.

SHEKEL, J.

ndefinite Admittance Representation of Linear Networks, Bull. Res. Counc. of Israel, 1954, 3, 390—394.

Kirchhoff's Rules, that form the basis of network analysis, apply to networks composed of two-terminal branches. In this paper, two postulates are proposed, that are applicable to networks whose basic elements may have more than two terminals (e.g., multi-electrode vacuum-tubes or transistors). The resulting method also enables analysis without specifying the voltage-reference terminal.

There is some redundancy in the definition of the currents, and excessive degrees of freedom in the voltages; but it is shown that these two effects compensate and result in no ambiguity in the definitions of instantaneous power and of the admittance matrix.

The various points raised in the paper are interpreted geometrically as relations in n-dimensional Euclidian space.

HAYMAN, H. J. G.

The Application of the Debye Charging Process to Polyions, Bull. Res. Counc. of Israel, 1954, 3, 395—397.

BIRK, M., EREZ, A., MANHEIMER, Y. and NAHMANI, G.

On Electrical Conductivity in Detonation and Shock Waves, and the Measurement of Detonation and Shock Velocities, Bull. Res. Counc. of Israel, 1954, 3, 398—413.

Electrical conduction was measured between electrodes on detonating explosives, and in shock waves in air. The current was found to increase with area and with voltage and to decrease when the gap was widened. The ratio of voltage to current, which was designated for convenience "ionization resistance", was practically constant in detonations, but increased slightly with voltage in shock waves. It also increased, exponentially, with the distance from the charge. Comparison of theory with results seems to indicate, that the ionized gases are not in a state of energy equilibrium. Velocities of detonations and of shock waves were measured by a system where pulses from probes were used for blanking the beam in an oscillograph with a spiral time base.

LANDSBERG, R. and SEIBALD, S.

Some Thermodynamic Properties of Freon-22, Bull. Res. Counc. Israel, 1953, 3, 414—416.

The specific heat of monochloro-difluoro-methane ($CHClF_2$; Freon-22) has been expressed as function of pressure and temperature.

By means of Clausius' equation, this leads to an expression for the specific volume. On this basis the isentropic exponent is determined and shown to be noticeably different from the ratio of the specific heats.

TAMARI, D.

On a Generalization of Uniform Structures and Spaces, Bull. Res. Counc. of Israel, 1954, 3, 417—428.
The introduction states the problem of asymmetrical directed space concepts corresponding to the irreversibility of nature. Using the language of binary relations we consider generalizations of the concept of uniform structures and define so called quasi-ordoform and ordoform structures. These structures generalize the order and quasi-order relations exactly in the same way as the uniform structures generalize the relation of equivalence. In particular, we distinguish also separate (Hausdorff) structures (generalizing in the same sense the relation of equality) for which "quasi-ordoform" coincides with "ordoform". From every structure a topology is deduced. It is easy to transfer the concepts and first theorems of "uniform continuity" to those of "(q)uasi)-ordoform continuity". Some concepts are illustrated by examples. The generalized problem of metrizability remains open.

BERGMANN, E. D.

Spectographic Observations on "Non-Alterna" Hydrocarbons, Bull. Res. Counc. of Israel, 1954, 3, 429—434
1) Fulvenes and to some extent also some other "non-alternant" hydrocarbons show the same bathochromic shift as open-chain polyenes when their spectra in carbon disulphide are compared with those in alcoholic solution.
2) In 9,12-diaryl-diphensuccindadienes (VIII, R=aryl), the aryl groups cause a strong bathochromic shift of the longest absorption band, in contradistinction with the behaviour of the analogously built substances of the 1-phenylnaphthalene type. This is in accord with the predictions of the molecular orbital theory.
3) In the benzofulvene series, substitution of hydrogen by halogen causes the same type of bathochromic shift that has been observed for azulene. Diphensuccindadiene does not show an analogous effect.
4) The spectra of a number of 2,3-diarylindones (X) have been determined. The two main absorption bands cannot be correlated to specific "absorption paths" in the molecule.

תמרי, ד., על אודות הפללת הסטרוקטורות והמרחבים האוניפורמיים 3, 417-428

הקדמתנו דנה בבעיה להתאים להתליכי־טבע בלתי־הפיכים (אירודיביליים) מושגי מרחב אסימטריים מכוונים. אנו מתכלים בעורת לשון תורת הרלציות הבינאריות, בהכללה מסוגנת של מושגי הסטרוקטורה האוניפורמית בהגדירנו סטרוקטורות קווי־אורדופורמיות בתוכן הסטרוקטורות האורדופורמיות. באותו המובן שאפשר להסתכל על הסטרוקטורות האוניפורמיות כהכללות של יחס (רלציה) אינדיוידואלי מסוים, כלומר יחס השקילות (אקווילנציה), כך אפשר להבין את הסטרוקטורות הקווי־אורדופורמיות והאורדופורמיות כהכללות יחס הקווי־סדר ויחס הסדר. מבחינים במיוחד גם בסטרוקטורות "מפרידות". כל סטרוקטורה קווי־אורדופורמית טפרידה היא מטילא אורדופורמית מפרידה. לכל סטרוקטורה מתאימה טופולוגיה נגזרת ממנה. אפשר גם להעביר מיד את מושגי הרציפות האוניפורמית וכמה ממשפטיה היסודיים למונחי הרציפות האורדופורמית. כמה מהטושנים החדשים טודגמים על ידי טשלים. בעית "הסטריוציה" המוכללת נשארת פתוחה.

ברגמן, א. ד., תצפיות ספקטרוגרפיות על פחמימנים "גון־אלטרנא" 3, 429-434

(1) הפולבינים ובמדה מסוימת גם כמה פחמימנים אחרים ששייכים לקבוצת הפחמימנים "גון־אלטרנא" מראים אותה התנוזה הבתוכרומית של הספקטרום כמו הפוליאינים הפתוחים, אם משוים תמיסותיהם הכוהליות עם אלו בדיסולפיד הפחמן.

(2) ב־9,12־דיאריל־דיפינילקצאינדאריאנים (VIII), נוכחות הקבוצות האריליות גורמת לתנוזה בתוכרומית נכרת של סם הבליעה הארוך ביותר. תופעה זו מנוגדת להתנהגות הפחמימנים מסוג ה־1־פניל־נפתלין, על אף הדמיון במבניהם.

(3) בשורת הבנזופולבינים, גורמת התמרת הטימן בהלוגן לאותו הסוג של אפקט בתוכרומי כמו בשורת האוולינים. דיפינלסוקצאינדאריאן אינו מראה תכונה דומה.

(4) נמדדו הספקטרא של מספר 2,3־דיאריל־אינדונים (X). אי אפשר ליחס שני פסי הבליעה העיקריים למנגנוני בליעה ספציפיים במולקולה.

לויצקי, י., משפטים אחדים על חוגי צורן 3, 380-384

חוג S נקרא חוג צורן מטפוס I אם בתוך זה כל אידיאל ימני לא גילי מכיל אבר אידמפוטנטי השונה מ־0. חוג מטפוס I אשר כל תמונותיו ההומומורפיות אף הן חוגים מטפוס I נקרא בשם חוג מטפוס FI. במאמר הנוכחי מסתמך המחבר על משפטים אחדים אשר הוכחו על ידו במאמר אחר שהופיע זה לא כבר (עיין מאמר (13) ברשימה הביבליוגרפית) וטסיק בעזרתם משפטים נוספים אחדים על מבנה החוגים מטפוס I, המהווים תכלה יברת של משפטים שהיו ידועים עד כה רק בתורות האלגבראיות הכופיות ובתורת החוגים המקיימים את תנאי המינימום.

אגמון, ש., על הנקודות הסינגולריות של מחלקת טורי דיריכלה 3, 385-389

העבודה דנה בכמה תכונות של נקודות הסינגולריות של פונקציה $f(s)$ הניתנת להצגה על ידי טור דיריכלה שסדרת מעריכיו $\{\lambda_n\}$ ממלאת את התנאי $\lim(\lambda_{n+1}-\lambda_n)>0$ אם הציר המדומה הנו ציר ההתכנסות ואם הנקודות הסינגולריות של $f(s)$ בקטע I של הציר המדומה שארכו חלוי בסדרה $\{\lambda_n\}$ הן קטבים פשוטים $i\alpha_q$ $(q=1,\ldots,k)$ מוכח שמבנה שאר הנקודות הסינגולריות על הציר המדומה נקבע במידה רבה על ידי הקטבים דלעיל. מתקבל, למשל, שכל נקודה סינגולרית אחרת $i\alpha$, מבודדת על הציר המדומה, הנה בהכרח קוטב פשוט. כן α הנו צרוף ליניארי בעל מקדמים שלמים של $\alpha_1,\ldots,\alpha_k$.

שקל, י., תאור רשתות ליניאריות על ידי הולכה בלתי מסוימת 3, 390-394

הזקי קירכהוף המהוים את היסוד לשיטות השונות של נתוח רשתות, מותאמים לרשתות מרכבות מענפים בעלי שני הדקים. במאמר זה מוצעים שני פוסטולאטים אשר עליהם אפשר לבסס נתוח רשתות שמרכיביהן היסודיים הם בעלי כל מספר הדקים רצוי (למשל, שפופרות אלקטרונניות או טרנסיסטורים). בשיטה זו אין צורך לתאר מרכיב כזה על ידי רשת של ענפים ומקורות בעלי שני הדקים, כנהוג בכל השיטות האחרות. שיטה זו גם מאפשרת נתוח רשתות מבלי לציין במפורש את צומת היחוס להגדרת המתחים.

השיטה מצטיינת בהגבלות מסויימות בהגדרת הזרמים, ולעומתן חפש יתר בהגדרת המתחים; אולם מתברר שאין זה גורר כל אי בהירות או רב־משמעיות בהגדרות המספק הרגעי ומטריצת ההולכה.

בסיום המאמר בא תאור נאומטרי של הנקודות השונות המתעוררות בשיטה המוצעת, ביחסים בין גדלים נאומטריים במרחב אוקלידי בן n ממדים.

היטן, צ., שטוש בתהליך הטעינה של דבקי לסולי־יונים 3, 395-397

בירק, א., ארז, א., מנהימר, י., ונחמני, ג. 3, 398-413

על מוליכות חשמלית בדטונציה ובגלי הלם, ומדידת מהירויות הדטונציה וההלם.

לנדסברג ר., וויבלד, ס., סנולות תרמודינמיות אחדות של פריאון־22 ... 3, 414-416

הושגה נוסחה של החום הסגולי של מונוכלורדיפלואורמתן ($CHClF_2$ פריאון — 22) בתור פונקציה של לחץ וטמפרטורה, וע"י טשואת קלאוסיום גם נוסחה לנפח הסגולי. על בסיס זה נקבע מעריך האיזנטרופיה, ונתברר שהוא שונה במידה ניכרת מיחס החתומים הסגולים.

לב, ז״, מסקנות הנזבעות מהנחה של רדיום גרעיני של $1{,}2\times10^{-13}$A$\frac{1}{2}$ 3, 337-347

הרדיום הגרעיני הוא פרמטר רגיש מאד לתשובים של מומנט קוודרופולי גרעיני, מעברי נמה, מצבים מעוררים של גרעינים זוגים־זוגים, והסטות איזוטופיות. המאמר מראה כי הרדיום המוקטן ל־$1{,}2\times10^{-13}$A$\frac{1}{2}$ נותן התאמה טובה יותר בין חנכיון להערכים המחושבים, המשטעות של חלוקה בלתי אחידה של מטען גרעיני נרונה.

אולנדורף, פ׳, על התאוצה המעוררת בקרינה של חלקיקים חשמליים ראשוניים 3, 348-358

ההתקנים לשם ההאצה של חלקיקי־חשמל ראשוניים מבוססים עד כה על השמוש של שדות־אלקטריים לונגיטודינליים שמהירותם הפאסית מסונכרנת למהירות הנופנית של החלקיקים. לעומת זאת נותן השמוש של לחץ הקרינה המלווה נלום אלקטרומנגנטים טרנסורסליים את האפשרות של הנעה אסינכרונית. הנוסח המתמטי של פעולה זו דורש את החצבה של כח המבטא את תנובת הקרינה לתנועה. הסכימה של משוואות התנועה המושלמות בדרך זו ניתנת הן עבור המקרה של החלקיק הנח והן עבור המקרה של החלקיק הנע. בתנאים ארציים יוצא לחץ הקרינה בעצמה כה זעומה שאינו מתאים למטרות טכניות, ברם יש ביכולתו להעביר לחלקיק המואץ כמויות אנרניה ניכרות בפעלו לארך דרכים בנות מידות אסטרונומיות, על ידי התרכבה של שדה מנגנט מתמיד לשדה האלקטרודינמי מצליחים אנחנו להגביר את לחץ הקרינה פי כמה וכמה כשמביאים את התנופה המאולצת על ידי השדה המנגנטי המתמיד לידי תהודה לתדירות של הגל המתקדם, ויש ועל ידי בחירה מתאימה של תדירות זו מתגשמת התתהודה אב־טומטית באמצעות האפקט הרלטיביסטי לדופלר החל ביחס לחלקיק המונע. לאור התוצאות האלה מסתבר שיתנתן לבניה טכונה כעין סיקלוטרון שיהיה ביכולתה להשלים את מאיצי־החלקיקים הידועים עד כה, אותו התהליך פותח לנו הסבר חדש למבניהם הקרינה הקוסמית לאמר שבהן ישנם חלקיקים אשר יצאו מחלקי הספירות של כוכבים בהם שוררים שדות מנגנטיים חזקים כדוגמת בתמי השמש. אולם במקרה זה יש לצפות לתופעות־תהודה מטושטשות בלבד אשר נסותן הכמותי יהדוש שקולים סטטיסטיים.

דה־שליט ע״, תקיפת הסכוב בשדות לא־ספריים 3, 359-363

נוספת בור לאופרטור תקיפת הסכוב של הטפה מתקבלת נם לשדות לא־ספריים כלליים. יחסי החלוף של אופרטורי השדה ומכנה האופרטור של תקיפת הסכוב נובעים מדרישת האינוריאנטיות של האינטראקציה לנבי סבובים במרחב.

ארדש, פ״, דבורצקי, א״, קקוטני, ש״, נקודות רבוי של מסילות התנועה הבראונית במישור 3, 364-371

המאמר מטפל במסילות של תנועת־בראון (המתימטית) במישור, וחוקר את מציאותן של נקודות רבוי מסדר k, ז״א, נקודות שהמסילה עוברת דרכן k פעמים. התוצאה העיקרית היא כי כמעט לכל המסילות, ז״א, לקבוצת מסילות שהמתברותה אפס, יש נקודות־רבוי מכל סדר סופי שהוא.

ריינר, מ״, תופעות משניות באלסטיות והדרודינמיקה 3, 372-379

תופעות משניות מופיעות באלסטיות קוו־ליניארית כאשר העוות הוא סופי. הן מתגלות בגלל הדרישה שדחיה פשוטה אפשרית רק כאשר קיים (1) מתיחה או לחץ איזוטרופי (תופעת קלוין). (2) מאמצים מצטלבים דלקמן: מתיחה בכוון חנדריגט שלה (תופעת פוינטינג). שתי תופעות אלה מדנמות בפתול פשוט. אבר האלסטיות המצטלבת בחוק הכללי של אלסטיות איזוטרופית מרשה הופעת מאמצים מצטלבים בעלי סימן כל שהו. אין תופעה משכית מסוג זה קיימת בצמיגות קוויניאריות. אבר האלסטיות המצטלבת בחוק הצמיגות הכללי מגלה קיום של מאמצים מצטלבים בעלי אותו סימן. דבר זה מדגם בתופעות המשאבה הצנטריפוגלית.

כהן, ש. ושמורק, מ., נסיון לחבחין בתנועה אלקטרוליטית של יוני כסף רדיואקטיביים הנוצרים בין נקודת השריג של גבישי כלוריד הכסף כתוצאה מהקרנת הגבישים על ידי נויטרונים אטיים 3, 312-315

הקרנת גבישים יונים על ידי נויטרונים תרמיים צריכה לגרום ליצירת יונים רדיואקטיביים המוזזים ממקומם הנכון בשריג, על ידי רתיעה נרעינית בתהליך הדומה ל"אפקט של זילרד-צ'לי" פורם", בתנאים מסוימים, אפשר לצפות לגלות את תנועת היונים הרדיואקטיביים האלה בשדה חשמלי, על ידי שטות של עקבות רדיואקטיביות. נסיון נעשה לגלות את תנועת יוני הכסף הרדיואקטיבי בגושי כלוריד הכסף המוקרנים על ידי נויטרונים תרמיים, בהשפעת שדה מתחנואות השלילית יש להסיק שבטמפרטורה של 80— מעלות צלזיוס, המובילות של יוני הכסף המוזזים בשריג היא קטנה מ־$7\times10^{-8}cm^2/volt\,sec$ והאנרגיה של אקטיבציה עבור דיפוזיה גדולה מ־0.23e.v.

יפה, א. א., קרני־X מקלפת M של רדיום D ונצולת הפלואורסצנציה מקרינת M של בריקת 3,

בדיקת הספקטרום הרך של רדיום D ער לאנרגיה של 1 Kev, בעזרת מונה פרופורציונלי הנבנה בכיוחד לצורך זה, מראה כי קיימת בו קרינת M, עצמת הקרינה ממקור זה נמדדה ביחס לעצמת קרינת L מאותו המקור.

על בסיס מערכת ההתפוררות של רדיום D והתחליכים הבאים בעקבות היוניזציה של רדיום E על ידי קרינת L, אפשר להראות כי המידה בה מרשה את הערכת ניצולת הפלואורסצנציה של קרינת M של ביסמוט. הנצולת השלמה של רדיום E (ביסמוט) תיא 0.007 ± 0.0037.

מיזם, ה., הטפול במקרי "תיקו" במתאם־דרגות מנקודת ראות של תורת ההסתברות 3, 321-327

מתוארת ההתפלגות של מקדמי "מתאם־דרגות" המתקבלים בחשוב מקדם המתאם בשביל כל הזוגות של מסדרים שבהם מופיעים מקרי תיקו המתקבלים על ידי הכנסת מקרי תיקו בכל האפנים השונים האפשריים לתוך זוג מסוים של מסדרים שבהם אין מקרי תיקו. התפלגות זו הנה א־סימפטרית והממוצע האריתמטי שלה שונה מהקדם דרגות של הזוג הנדון של דרגות חסר מקרי תיקו.

רוזן, נ., גלי־נרביטציה גליליים 3, 328-332

במאמר נתנים פתרונות אחדים של משוואות שדה הגרביטציה בתנאי הסימטריות הכלליות, בצורת גלים גליליים.

נמצא כי קיימת אפשרות לקבל, בתאור גלים כאלה, פתרונות ללא סינגולריות.

פקטה, מ., על הרציפות למחצה של הקוטר הטרנספיניטי 3, 333-336

הקוטר הטרנספיניטי $\tau(s)$ של קבוצה אין סופית קומפקטית s של נקודות בטישור ה־z הטרכב היא פונקציית־קבוצה אשר תכונותיה נחקרו רבות מאז הפונקציה הזאת הוגדרה לפני כ־30 שנה על ידי מחבר העבודה הזו, התורמת תרומה חדשה להקירה השתנותו של $\tau(s)$ כאשר s משתנה באופן רציף.

העבודה שהקוטר הטרנספיניטי של סביבת ה־ρ, $s(\rho)$, של קבוצה שרירותית s שואף ל־$\tau(s)$ כשה־ρ שואף לאפס, היא הוצאה פשוטה של ההגדרות המקודמות הידועות.

הטפול בבעיה המקבילה של הרציפות למחצה שהוכרנה זה עתה, תופעה שקשה יותר להבחירה, היא מטרת המאמר העוכחי. נסתכל בטפוסים אחדים של קבוצות קומפקטיות s — אשר תתוארנה בפרוטרוט להלן (ראה משפטים I ו־II) — במטרה להראות את אפשרות מיצוין על ידי למניסקטות L כך שהמנה $\tau(L)/\tau(s)$ תתקרב ליחידה כרצוננו.

ילקוט המועצה המדעית לישראל

כרך III | מרץ 1954 (הוצ"ל יוני) | מס. 4

פרנקל, א' ח', התפיסה האינטואיציונית במתימטיקה ותורת ההגיון 3, 283–289

תאור לתפיסתם החדישה של מהות המתימטיקה ויסודותיה, המכונה ניו־אינטואיציוניזם על ידי מיסדה ברואר, על קויה האופייניים נמנים: זהוי של "מציאות" במתמטיקה עם בניה; שלילת העקרון ההיזוני של "השלישי־הנמנע"; ותור על מושג ברציפות במובנה של המתמטיקה הקלסית. קבלת התפיסה הזאת על ידי פיסיקאים תלויה בהמצאת שטות חדשות באנליזה המתמטית.

רקח, י', ההתחרות בין הקונפיגורציות הנמוכות בספקטרים של שורת הברזל 3, 290–298

נתונות נוסחאות אינטרפולציה לשם חשוב הגובה היחסי של רמות האנרגיה בעלות כפילות עליונה בקונפיגורציות הנמוכות של הספקטרים של שורת ברזל. נידונים גם שטושים לקונפיגורציות אחרות ולשורות אחרות של המערכת המחזורית.

טיטלבוים, פ', מומנטים מגנטיים בגרעינים אי זוגיים ב־jj-coupling 3, 299–303

טפול במומנטים מגנטיים בגרעינים המכילים חלקיקים אקויולנטים מחוץ לקליפות הסגורות בהנחה שלכל חלקיק בקליפה החיצונית יש מומנט מגנטי השוה לערך של החלקיק יחיד בריאגרטה של שמידט. אבל בצרפנו את כל החלקיקים בין פרוטונים ובין נויטרונים למצבים המאופיינים על ידי הספין האיזוטופי T, תקופת הסבוב הכללית J ותק s וספין איזוטופי מצומצם t, על ידי שמוש בקואפיצינטים של אבאות חלקית וחשובם מקבלים את הנוסחה של מומנטים מגנטים בשביל המצבים $J=l$ וזתק $s=1$. מתוך הנוסחה רואים שבדרך כלל המומנטים המגנטים יפלו בין קוי שמידט.

פוקס, ד', התהוות אנטיפרוטון ע"י התנגשות שני פרוטונים 3, 304–307

התחך הפעיל של התהוות זוג של פרוטון־אנטיפרוטון על ידי התנגשות של שני פרוטונים מחושב בסביבתו של ערך־הסף בשמוש בשיטת פינמן ובהנחת מסונים פסידו־סקלריים בתצמדה פסידו וקטוריאלית. המחבר מוכיח שאפשר להעזר בנמוכי סימטריה, מבלי להוציא את החשובים לפועל, לקבלת מושג על התכונות הזויתיות והאנרגטיות של התהליך. נמוקים אלה יכולים לשמש גם עבור תאוריות אחרות של הצמדה רופפת, תגדונות במאמר. התוצאות טושות לאלה שהושגו ע"י פרמי וע"י טקטני ומשירה.

נוימן, י', על הגל החרטוני השני במהלך השנתי של קרינת השמש ותוצאת אפסיו 3, 308–311

צורה פשוטה באופן יחסי ניתנת לגל החצי־שנתי של קרינת השמש המגיעה אל פסגת האטמוספירה. הפיתוח מתבסס על הנחות בעלות אופי קל. חקירת תכונות גל זה מראה, בין היתר, כי אמפלי־טודת הגל מתאפסת ברחבים הבינוניים של כל אחת משתי המחציות של כדור הארץ. סורדרוס מצא כי המהלך השנתי של ההתארות מהאוקיאנוסים ברחבים הבינוניים הוא גל כפול: תכונות הגל החצי־שנתי, בשלוב עם תכונות הולכת החום בתוך גופי מים, מסבירות את תוצאותיו של סורדרוס.

המועצה המדעית לישראל

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